Answers to Additional Exercises in *Generalizability Theory*

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Chapter 1

1.3 According to the Spearman-Brown Formula, reliability for a measurement procedure with twice the number of prompts (i.e., with six prompts) is $2(.74)/(1+.74) = .85$. According to generalizability theory, however,

$$\begin{aligned}
\hat\sigma^2(\delta) &= \hat\sigma^2(pT) + \hat\sigma^2(pR) + \hat\sigma^2(pTR) \\
&= \frac{\hat\sigma^2(pt)}{n'_t} + \frac{\hat\sigma^2(pr)}{n'_r} + \frac{\hat\sigma^2(ptr)}{n'_t n'_r} \\
&= \frac{.15}{6} + \frac{.04}{2} + \frac{.12}{12} \\
&= .055,
\end{aligned}$$

and it follows that

$$\boldsymbol{E}\hat\rho^2 = \frac{\hat\sigma^2(p)}{\hat\sigma^2(p) + \hat\sigma^2(\delta)} = \frac{.250}{.250 + .055} = .82.$$

The explanation for this difference is that the term $\hat\sigma^2(pR)$ in $\hat\sigma^2(\delta)$ is unaffected by doubling the number of prompts, whereas the Spearman-Brown procedure effectively divides $\hat\sigma^2(pR)$ by two. This is an illustration of the fact that the error term is undifferentiated in classical theory, whereas generalizability theory can take into account the relative contributions of different numbers of prompts and raters to error variance.

Chapter 2

2.2 The ANOVA table is

Effect(α)	$df(\alpha)$	$SS(\alpha)$	$MS(\alpha)$	$\hat{\sigma}^2(\alpha)$
r	4	.700	.175	.0012
i	125	7.144	.057	.0061
ri	500	13.353	.027	.0267

Using Equation 2.38 the standard error of $\overline{X}$ is

$$\hat{\sigma}(\overline{X}) = \sqrt{\frac{.0012}{5} + \frac{.0061}{126} + \frac{.0267}{5 \times 126}} = .018.$$

2.4 The usual formula for KR–20 is

$$\text{KR–20} = \frac{n_i}{n_i - 1}\left[1 - \frac{\sum \overline{X}_i(1 - \overline{X}_i)}{n_i^2 S^2(p)}\right],$$

where $S^2(p) = \sum(\overline{X}_p - \overline{X})^2/n_p$. From the summary statistics reported in Table 2.2, it is easy to verify that $S^2(p) = .0612$ and $\sum \overline{X}_i(1 - \overline{X}_i) = 1.99$. It follows that

$$\text{KR–20} = \frac{12}{11}\left[1 - \frac{1.99}{144(.0612)}\right] = .844,$$

which is identical to $\boldsymbol{E}\hat{\rho}^2$ reported in Table 2.4.

In classical theory, the usual formula for error variance is

$$\sigma^2(E) = S^2(p)(1 - r),$$

where r is reliability. Replacing $S^2(p)$ with $Est[\boldsymbol{E}S^2(p)]$ and r with $\boldsymbol{E}\hat{\rho}^2$, we obtain

$$\hat{\sigma}^2(E) = .0680(1 - .844) = .0106,$$

which is identical to $\hat{\sigma}^2(\delta)$ reported in Table 2.4. Note that $S^2(p)$ in KR–20 is actually the so-called "biased" estimate of variance, whereas $Est[\boldsymbol{E}S^2(p)]$, which is used her to estimate $\sigma^2(E)$, is the unbiased estimate. This inconsistency can be circumvented, but doing so complicates the expression for KR–20.

2.6 The usual formula for KR–21 is

$$\text{KR–21} = \frac{n_i}{n_i - 1}\left[1 - \frac{\overline{X}(1 - \overline{X})}{n_i S^2(p)}\right],$$

where $S^2(p) = \sum(\overline{X}_p - \overline{X})^2/n_p$, which is .0612 for Synthetic Data No. 1. It follows that

$$\text{KR–21} = \frac{12}{11}\left[1 - \frac{.5583(1-.5583)}{12(.0612)}\right] = .725.$$

Using Equation 2.54,

$$\widehat{\Phi}(\lambda = \overline{X}) = \frac{.0574 - .0131}{.0574 - .0131 + .0169} = .724.$$

The difference of .001 is attributable solely to rounding error.

2.8 Recall that

$$X_{pi} = \mu + \nu_p + \nu_i + \nu_{pi}.$$

It follows that

$$\begin{aligned} \overline{X}_p &= \frac{1}{n_i}\sum_i(\mu + \nu_p + \nu_i + \nu_{pi}) \\ &= \mu + \nu_p + \frac{1}{n_i}\sum_i \nu_i + \frac{1}{n_i}\sum_i \nu_{pi} \\ &= \mu + \nu_p + \nu_I + \nu_{pI}. \end{aligned}$$

In a similar manner, it can be shown that

$$\overline{X} = \mu + \nu_P + \nu_I + \nu_{PI}.$$

Therefore,

$$\boldsymbol{E}MS(p) = \boldsymbol{E}\left[\frac{T(p) - T(\mu)}{n_p - 1}\right] = \frac{\boldsymbol{E}\,T(p) - \boldsymbol{E}\,T(\mu)}{n_p - 1},$$

where

$$\boldsymbol{E}\,T(p) = \boldsymbol{E}\left[n_i \sum_p \overline{X}_p^2\right] = n_i\,\boldsymbol{E}\left[\sum_p(\mu + \nu_p + \nu_I + \nu_{pI})^2\right]$$

and

$$\boldsymbol{E}\,T(\mu) = \boldsymbol{E}\left[n_p n_i \overline{X}^2\right] = n_p n_i\,\boldsymbol{E}\left[(\mu + \nu_P + \nu_I + \nu_{PI})^2\right].$$

Since all the covariances are zero,

$$\begin{aligned} \boldsymbol{E}\,T(p) &= n_i \sum_p(\mu^2 + \boldsymbol{E}\nu_p^2 + \boldsymbol{E}\nu_I^2 + \boldsymbol{E}\nu_{pI}^2) \\ &= n_p n_i\,\mu^2 + n_p n_i\,\sigma^2(p) + n_p n_i\,\boldsymbol{E}\nu_I^2 + n_p n_i\,\boldsymbol{E}\nu_{pI}^2. \end{aligned}$$

Now,

$$\boldsymbol{E}\nu_I^2 = \boldsymbol{E}\left(\frac{1}{n_i}\sum_i \nu_i\right)^2 = \frac{1}{n_i^2}\boldsymbol{E}\left(\sum_i \nu_i\right)^2 = \frac{1}{n_i^2}\left(n_i \boldsymbol{E}\nu_i^2\right) = \frac{\sigma^2(i)}{n_i}.$$

Similarly, it can be shown that $\boldsymbol{E}\nu_{pI}^2 = \sigma^2(pi)/n_i$, and, therefore,

$$\boldsymbol{E}\,T(p) = n_p n_i\,\mu^2 + n_p n_i\,\sigma^2(p) + n_p\,\sigma^2(i) + n_p\,\sigma^2(pi).$$

A corresponding proof gives

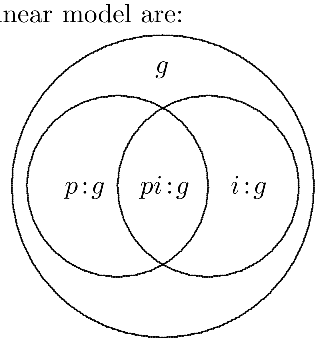
$$\boldsymbol{E}\,T(\mu) = n_p n_i\,\mu^2 + n_i\,\sigma^2(p) + n_p\,\sigma^2(i) + \sigma^2(pi).$$

It follows that

$$\begin{aligned}
\boldsymbol{E}\,MS(p) &= \frac{\boldsymbol{E}\,T(p) - \boldsymbol{E}\,T(\mu)}{n_p - 1} \\
&= \frac{[n_p n_i\,\sigma^2(p) - n_i\,\sigma^2(p)] + [n_p\,\sigma^2(pi) - \sigma^2(pi)]}{n_p - 1} \\
&= \frac{n_i(n_p - 1)\sigma^2(p) + (n_p - 1)\sigma^2(pi)}{n_p - 1} \\
&= n_i\,\sigma^2(p) + \sigma^2(pi).
\end{aligned}$$

Chapter 3

3.3 This is a multiple matrix sampling design with three matrices, each of which consists of 40 items administered to 100 students. These three designs are disconnected in the sense that they have no common items or students. Letting p be students, i be items, and g be student/item groups, the design structure is $(p \times i):g$, and the Venn diagram and linear model are:



$$\begin{aligned}
X_{pig} = \; & \mu \\
& + \nu_g \\
& + \nu_{p:g} \\
& + \nu_{i:g} \\
& + \nu_{pi:g}
\end{aligned}$$

The variance components are $\sigma^2(p)$, $\sigma^2(i)$, and $\sigma^2(pi)$. There is no variance component for groups—or, stated differently, since groups are formed randomly, $\sigma^2(g)$ is necessarily zero. From either point of view, groups have no bearing on the population or universe. The simplest way to estimate the variance components is to conduct three $p \times i$ analyses and average the three sets of estimated variance components.

3.6

Effect(α)	$df(\alpha)$	$T(\alpha)$	$SS(\alpha)$	$MS(\alpha)$	$\hat{\sigma}^2(\alpha)$
p	9	390.6667	15.6667	1.7407	.1148
t	2	413.1000	38.1000	19.0500	.8556
r	1	378.2667	3.2667	3.2667	.0667
pt	18	445.0000	16.2333	.9019	.4111
pr	9	396.0000	2.0667	.2296	.0500
tr	2	418.6000	2.2333	1.1167	.1037
ptr	18	454.0000	1.4333	.0796	.0796

3.8 (a) The design is $(p{:}c) \times i$. (b) The following class-by-item means can be used to compute the ANOVA results provided subsequently:

	Class-by-Item Means												
Class	1	2	3	4	5	6	7	8	9	10	11	12	$\overline{X}_c$
1	1.0	.8	.8	.6	.6	.2	.2	.2	.0	.0	.0	.0	.367
2	1.0	1.0	1.0	.8	.8	1.0	1.0	.6	.6	.6	.4	.2	.750

Effect	$df(\alpha)$	$T(\alpha)$	$SS(\alpha)$	$MS(\alpha)$	$\hat{\sigma}^2(\alpha)$
c	1	41.8167	4.4083	4.4083	.0665
$p{:}c$	8	44.7500	2.9333	.3667	.0205
i	11	47.1000	9.6917	.8811	.0709
ci	11	53.4000	1.8916	.1720	.0102
$pi{:}c$	88	67.0000	10.6667	.1212	.1212
Mean(μ)		37.4083			

(c) $\hat{\sigma}^2(i|I)$, $\hat{\sigma}^2(ci|I)$, and $\hat{\sigma}^2(pi{:}c|I)$, are the same as for the random model, although strictly speaking, $\hat{\sigma}^2(i|I)$ is a quadratic form. The two estimates that do change are:

$$\hat{\sigma}^2(c|I) = \hat{\sigma}^2(c) + \frac{\hat{\sigma}^2(ci)}{n_i} = .0665 + \frac{.0102}{12} = .0674,$$

and

$$\hat{\sigma}^2(p{:}c|I) = \hat{\sigma}^2(p{:}c) + \frac{\hat{\sigma}^2(pi{:}c)}{n_i} = .0205 + \frac{.1212}{12} = .0305.$$

(d) Let M stand for the model. Then, $\hat\sigma^2(p{:}c|M)$ and $\hat\sigma^2(pi{:}c|M)$ are the same as for the random model. The other estimated variance components are:

$$\hat\sigma^2(c|M) = \hat\sigma^2(c) + \frac{\hat\sigma^2(p{:}c)}{N_p} = .0665 + \frac{.0205}{30} = .0672,$$

$$\begin{aligned}\hat\sigma^2(i|M) &= \hat\sigma^2(c) + \frac{\hat\sigma^2(ci)}{N_c} + \frac{\hat\sigma^2(p{:}c)}{N_c N_p}\\ &= .0709 + \frac{.0102}{10} + \frac{.1212}{(10)(30)} = .0723,\end{aligned}$$

and

$$\hat\sigma^2(ci|M) = \hat\sigma^2(ci) + \frac{\hat\sigma^2(pi{:}c)}{N_p} = .0102 + \frac{.1212}{30} = .0142.$$

Chapter 4

4.2 The manner in which score effects are defined guarantees that $\boldsymbol{E}_p(\nu_{pH}) = \boldsymbol{E}_H(\nu_{pH}) = \boldsymbol{E}_H(\nu_H) = 0$. Now,

$$\begin{aligned}\sigma(\nu_{pH}, \nu_H) &= \underset{p}{\boldsymbol{E}}\,\underset{H}{\boldsymbol{E}}\,\nu_{pH}\nu_H - \underset{p}{\boldsymbol{E}}\,\underset{H}{\boldsymbol{E}}\,\nu_{pH}\,\underset{H}{\boldsymbol{E}}\,\nu_H\\ &= \underset{H}{\boldsymbol{E}}\,\nu_H\,\underset{p}{\boldsymbol{E}}\,\nu_{pH}\\ &= 0.\end{aligned}$$

4.3 The linear model for the $p \times (I{:}H)$ design in Equation 4.2 is

$$\overline{X}_p = \mu + \nu_p + \nu_H + \nu_{I:H} + \nu_{pH} + \nu_{pI:H}.$$

It follows that the mean for the group of n'_p persons, P, is

$$\overline{X} = \mu + \nu_P + \nu_H + \nu_{I:H} + \nu_{PH} + \nu_{PI:H}.$$

Therefore, letting $\mathcal{R}$ be I and H,

$$\begin{aligned}\sigma^2(\overline{X}) &\equiv \underset{\mathcal{R}}{\boldsymbol{E}}\,\underset{P}{\boldsymbol{E}}(X_{P\mathcal{R}} - \mu)^2\\ &= \underset{H}{\boldsymbol{E}}\,\underset{I}{\boldsymbol{E}}\,\underset{P}{\boldsymbol{E}}(\nu_P + \nu_H + \nu_{I:H} + \nu_{PH} + \nu_{PI:H})^2\\ &= \underset{P}{\boldsymbol{E}}\,\nu_P^2 + \underset{H}{\boldsymbol{E}}\,\nu_H^2 + \underset{H}{\boldsymbol{E}}\,\underset{I}{\boldsymbol{E}}\,\nu_{I:H}^2 + \underset{P}{\boldsymbol{E}}\,\underset{H}{\boldsymbol{E}}\,\nu_{PH}^2 + \underset{P}{\boldsymbol{E}}\,\underset{H}{\boldsymbol{E}}\,\underset{I}{\boldsymbol{E}}\,\nu_{PI:H}^2\end{aligned}$$

because effects are uncorrelated. By the definition of the variance components, this gives

$$\begin{aligned}\sigma^2(\overline{X}) &= \sigma^2(P) + \sigma^2(H) + \sigma^2(I{:}H) + \sigma^2(PH) + \sigma^2(PI{:}H)\\ &= \frac{\sigma^2(p)}{n'_p} + \frac{\sigma^2(h)}{n'_h} + \frac{\sigma^2(i{:}h)}{n'_i n'_h} + \frac{\sigma^2(ph)}{n'_p n'_h} + \frac{\sigma^2(pi{:}h)}{n'_p n'_i n'_h}\\ &= \frac{\sigma^2(p) + \sigma^2(pH) + \sigma^2(pI{:}H)}{n'_p} + [\sigma^2(H) + \sigma^2(I{:}H)],\end{aligned}$$

which is in the form of Equation 4.20.

4.5 (a) For the $p \times T \times R$ design,

$\hat{\sigma}^2(\alpha)$	D Studies		
	n'_t	3	6
	n'_r	2	1
$\hat{\sigma}^2(p) = .1148$	$\hat{\sigma}^2(p)$	.115	.115
$\hat{\sigma}^2(t) = .8556$	$\hat{\sigma}^2(T)$	.285	.143
$\hat{\sigma}^2(r) = .0667$	$\hat{\sigma}^2(R)$	.033	.067
$\hat{\sigma}^2(pt) = .4111$	$\hat{\sigma}^2(pT)$	.137	.069
$\hat{\sigma}^2(pr) = .0500$	$\hat{\sigma}^2(pR)$	.025	.050
$\hat{\sigma}^2(tr) = .1037$	$\hat{\sigma}^2(TR)$	.017	.017
$\hat{\sigma}^2(ptr) = .0796$	$\hat{\sigma}^2(pTR)$	.013	.013
	$\hat{\sigma}^2(\tau)$	.115	.115
	$\hat{\sigma}^2(\delta)$	.175	.132
	$\hat{\sigma}^2(\Delta)$	.510	.359
	$\boldsymbol{E}\hat{\rho}^2$	.40	.47
	$\widehat{\Phi}$	.18	.24

(b) Since $\hat{\sigma}^2(t)$ is substantially larger than $\hat{\sigma}^2(r)$, and $\hat{\sigma}^2(pt)$ is substantially larger than $\hat{\sigma}^2(pr)$, using many more tasks than raters in the D study leads to lower error variances.

(c) For the $p \times i$ design, $\hat{\sigma}^2(p) = .2170$, $\hat{\sigma}^2(i) = .8282$, and $\hat{\sigma}^2(pi) = .4385$. Cronbach's alpha is the generalizability coefficient with $n'_i = 6$:

$$\boldsymbol{E}\hat{\rho}^2 = \frac{\hat{\sigma}_1^2(p)}{\hat{\sigma}_1^2(p) + \hat{\sigma}_1^2(pi)/n'_i} = \frac{.2107}{.2107 + .4385/6} = .742,$$

where the subscript "1" is used to emphasize that these are estimated variance components for the single-faceted universe.

(d) In terms of parameters,

$$\sigma_1^2(pi) = \sigma^2(pt) + \sigma^2(pr) + \sigma^2(ptr),$$

where variance components to the right are for the $p \times t \times r$ design. It follows that

$$\sigma_1^2(\delta) = \frac{\sigma_1^2(pi)}{6} = \frac{\sigma^2(pt) + \sigma^2(pr) + \sigma^2(ptr)}{6},$$

which is smaller than $\sigma^2(\delta)$ for the $p \times T \times R$ design:

$$\sigma^2(\delta) = \frac{\sigma^2(pt)}{3} + \frac{\sigma^2(pr)}{2} + \frac{\sigma^2(ptr)}{6}.$$

Therefore, the generalizability coefficient for the random effects $p \times I$ design is larger than the the generalizability coefficient for the random effects $p \times T \times R$ design.

4.8 The D study results are:

Design	$\hat{\sigma}^2(\tau)$	$\hat{\sigma}^2(\delta)$	$\hat{\sigma}^2(\Delta)$	$\boldsymbol{E}\hat{\rho}^2$	$\widehat{\Phi}$
$p \times T \times R$	.397	.030	.033	.931	.923
$p \times (R{:}T)$	.397	.030	.031	.931	.928

These results are very different from those for the random model because $\hat{\sigma}^2(pt)$ is very large. For the mixed model, $\sigma^2(pt)$ contributes to universe score variance, and it does not contribute to error variances.

4.11 Since

$$\begin{aligned} \boldsymbol{E}MS(c) &= \sigma^2(pi{:}c) + n_i\sigma^2(p{:}c) + n_p\sigma^2(ci) + n_p n_i \sigma^2(c) \\ \boldsymbol{E}MS(p{:}c) &= \sigma^2(pi{:}c) + n_i\sigma^2(p{:}c), \end{aligned}$$

it follows from Equations 4.33 that

$$\boldsymbol{E}\hat{\rho}^2(P) = \frac{MS(c) - MS(p{:}c)}{MS(c)} = 1 - \frac{MS(p{:}c)}{MS(c)}.$$

4.12 The signal, or universe score variance, is

$$\hat{\sigma}^2(\tau) = .03 + \frac{.17}{n'_p}.$$

The noise, or relative error variance, is

$$\hat{\sigma}^2(\delta) = \frac{.25}{8} + \frac{.28}{8\,n'_p} = .0063 + \frac{.0350}{n'_p}.$$

It follows that $\hat{\sigma}^2(\tau) \doteq 4.8\,\hat{\sigma}^2(\delta)$ for all values of n'_p, which means that $\boldsymbol{E}\hat{\rho}^2$ must be nearly constant for all n'_p.

4.13 (a) The estimated G study variance components clearly indicate that for both systolic and diastolic blood pressure, the day facet contributes much more to variability than the replication facet. Note especially that for both systolic and diastolic blood pressure, $\hat{\sigma}^2(pd)$ is over twice as large as $\hat{\sigma}^2(pr{:}d)$.

(b) With $n'_r = 1$ it is easily verified that, for systolic blood pressure,

$$\begin{aligned} \hat{\sigma}^2(\Delta) &= \frac{\hat{\sigma}^2(d)}{2} + \frac{\hat{\sigma}^2(r{:}d)}{2} + \frac{\hat{\sigma}^2(pd)}{2} + \frac{\hat{\sigma}^2(pr{:}d)}{2} \\ &= .365 + .150 + 12.455 + 4.955 \\ &= 17.925, \end{aligned}$$

$\hat{\sigma}(\Delta) = 4.2$, and

$$\widehat{\Phi} = \frac{\hat{\sigma}^2(p)}{\hat{\sigma}^2(p) + \hat{\sigma}^2(\Delta)} = \frac{83.41}{83.41 + 17.93} = .82.$$

(c) Since $\hat{\sigma}^2(d) = \hat{\sigma}^2(r{:}d) = 0$, the required inequality is

$$\widehat{\Phi} = \frac{\hat{\sigma}^2(p)}{\hat{\sigma}^2(p) + \hat{\sigma}^2(pd)/2 + \hat{\sigma}^2(pr{:}d)/2n'_r} > .8;$$

but even for $n'_r \to \infty$,

$$\widehat{\Phi} = \frac{\hat{\sigma}^2(p)}{\hat{\sigma}^2(p) + \hat{\sigma}^2(pd)/2} = \frac{36.49}{36.49 + 21.69/2} = .77.$$

Therefore, with $n'_d = 2$, there is no value of n'_r such that $\widehat{\Phi} > .8$.

(d) In the universe of generalization, days is a fixed facet with $n'_d = 1$. It follows that, for systolic blood pressure,

$$\hat{\sigma}^2(\tau) = \hat{\sigma}^2(p) + \hat{\sigma}^2(p{:}d) = 83.41 + 24.91 = 108.32,$$

$$\hat{\sigma}^2(\Delta) = \hat{\sigma}^2(r{:}d) + \hat{\sigma}^2(pr{:}d) = .30 + 9.91 = 10.21,$$

$\hat{\sigma}(\Delta) = 3.2$, and

$$\widehat{\Phi} = \frac{108.32}{108.32 + 10.21} = .91.$$

For diastolic blood pressure,

$$\hat{\sigma}^2(\tau) = 58.18, \qquad \hat{\sigma}^2(\Delta) = 7.99,$$
$$\widehat{\Phi} = .88, \qquad \text{and} \qquad \hat{\sigma}(\Delta) = 2.8.$$

Chapter 5

5.4 These results are easily confirmed with simple arithmetic. Note that

$$\overline{\sigma}^2_r(p|H) = .0411, \quad \overline{\sigma}^2_r(h|H) = .0024, \quad \overline{\sigma}^2_r(i{:}h|H) = .0245,$$
$$\overline{\sigma}^2_r(ph|H) = .0051, \qquad \text{and} \qquad \overline{\sigma}^2_r(pi{:}h|H) = .1500.$$

5.5 Since r and h are both fixed, the model restrictions are

$$\textstyle\sum_r \nu_r = \sum_r \nu_{rh} = \sum_r \nu_{ri:h} = 0$$

and

$$\textstyle\sum_h \nu_h = \sum_h \nu_{rh} = \sum_h \nu_{ph:r} = 0.$$

It follows that the linear model for the $(p{:}r) \times (I{:}H)$ design can be represented as:

$$X_{pI:rH} = \mu + \nu_r + \nu_{p:r} + \nu_{I:H} + \nu_{rI:H} + \nu_{pI:rH}.$$

Since universe score for an object of measurement is

$$\tau \equiv \underset{I}{\boldsymbol{E}}\, X_{pI:rH} = \mu + \nu_r + \nu_{p:r} = \mu_{p:r},$$

universe score variance as defined in Equation 5.18 is

$$\begin{aligned}\sigma^2(\tau) &\equiv \sum_r \boldsymbol{E}(\mu_{p:r} - \mu)^2/n_r \\ &= \sum_r \boldsymbol{E}(\nu_r + \nu_{p:r})^2/n_r \\ &= \sum_r \nu_r^2/n_r + \sum_r \boldsymbol{E}\nu_{p:r}^2/n_r \\ &= \frac{n_r - 1}{n_r}\sigma^2(r|RH) + \sigma^2(p{:}r|RH),\end{aligned}$$

given the Cornfield and Tukey definitions in Table 5.6.

5.6 For a randomly selected region, $\hat{\sigma}^2(\tau) = \hat{\sigma}^2(p|H) = .0411$. Neglecting the index r, and using Equation 5.1 as well as Rule 5.1.3, gives

$$\begin{aligned}\hat{\sigma}^2(\delta) &= \left(\frac{1 - n'_h/N'_h}{n'_h}\right)\hat{\sigma}^2(ph{:}r|RH) + \frac{\hat{\sigma}^2(pi{:}rh|RH)}{n'_i n'_h} \\ &= 0 + .1500/40 = .0038,\end{aligned}$$

which leads to $\boldsymbol{E}\hat{\rho}^2 = .0411/(.0411 + .0038) = .915$.

5.8 It is evident from Equations 5.26 and 5.27 that $\sigma^2(\delta_g) > \sigma^2(\delta_p)$ occurs when

$$\sigma^2(p{:}g)/n_p + \sigma^2(pI{:}g)/n_p > \sigma^2(pI{:}g),$$

which is algebraically equivalent to

$$\frac{\sigma^2(p{:}g)}{\sigma^2(pI{:}g)} > n_p - 1.$$

The left side is a signal-noise ratio, $S/N(\delta)$. As discussed in Section 4.1.4, the relationship between a signal-noise ratio and a generalizability coefficient is

$$\boldsymbol{E}\rho^2 = \frac{S/N(\delta)}{1 + S/N(\delta)}.$$

It follows that

$$\boldsymbol{E}\rho^2_{p:g} = \frac{\sigma^2(p{:}g)}{\sigma^2(p{:}g) + \sigma^2(pI{:}g)} > \frac{n_p - 1}{(n_p - 1) + 1} = \frac{n_p - 1}{n_p}.$$

5.10 The quadratic fit gives

$$\hat{\sigma}(\delta_p) = \sqrt{.00075 + .02063\,(.6) - .02101\,(.6)^2} = .075.$$

Now,

$$\hat{\sigma}^2(I) = \hat{\sigma}^2(\Delta) - \hat{\sigma}^2(\delta) = .00514 - .00475 = .00039.$$

Therefore, using Equation 5.38,

$$\hat{\sigma}(\delta_p) = \sqrt{\frac{.6(1-.6)}{40-1} - .00039} = .076.$$

5.11 Since there are only two occasions, there is only one covariance. Using the two columns under $\overline{X}_{po}$ in Table 3.1, the observed covariance is

$$\frac{10}{9}\left[\frac{284.125}{10} - (5.075)(5.475)\right] = .697,$$

which is identical to

$$\hat{\sigma}^2(p) + \hat{\sigma}^2(pI) = .5528 + \frac{.5750}{4} = .697.$$

Chapter 6

6.3 Recall that $\hat{\sigma}^2(p) = (M_p - M_{pi})/n_i$. For the synthetic data, $M_p = 10.2963$, $M_{pi} = 2.7872$, and $n_i = 12$. Therefore, $\hat{\sigma}^2(p) = .6258$, and $k_p = k_{pi} = 1/12$. Note that k_{pi} is *not* $-1/12$; the k coefficients themselves are all positive, even when they are associated with subtracting a mean square to obtain $\hat{\psi}$.

Recall that, in the Burdick and Graybill (1992) notational conventions used in Section 6.2.3, $F_{\alpha:\eta_1,\eta_2}$ is the percentile point such that the area to the *right* is α. For the example, $\eta_1 = df(p) = 9$, $\eta_2 = df(pi) = 99$, and the required percentile points are:

$$F_{.1:9,\infty} = 1.6315, \quad F_{.9:99,\infty} = .8227, \quad \text{and} \quad F_{.1:9,99} = 1.6956,$$

which lead to

$$G_1 = .3871, \quad H_2 = .2155, \quad \text{and} \quad G_{12} = .0039,$$

respectively. Using these values, we obtain $V_L = .1136$, $\sqrt{V_L} = .3370$, and

$$\hat{\sigma}^2(p) - \sqrt{V_L} = .6258 - .3370 = .289.$$

6.4 In general, given estimated variance components for a $p \times i$ design, $\hat{\sigma}^2(\Delta)$ for the $I\!:\!p$ design is identical to that for the $p \times I$ design. A similar statement holds for $\hat{\sigma}[\hat{\sigma}^2(\Delta)]$ in Equation 6.29. Since the confidence intervals in Table 6.5 are for $\sigma^2(\Delta)$ with $n'_i = 12$, doubling the limits gives the confidence intervals for $\sigma^2(\Delta)$ for the $I\!:\!p$ design with $n'_i = 6$ in the mean score metric. Taking the square root of these limits gives the limits for $\sigma(\Delta)$. So, for example, the Satterthwaite interval for $\sigma^2(\Delta)$ in the mean score metric is (.503, .771), which leads to (.709, .878) as the interval for $\sigma(\Delta)$. Multiplying these limits by $n'_i = 6$ gives limits for $\sigma(\Delta)$ in the total score metric—(4.25, 5.27) for the Satterthwaite procedure.

6.7 Using the notational conventions for mean squares in Section 6.3.2, the signal-noise ratio for a single item is

$$\zeta = \frac{(\boldsymbol{E}M_p - \boldsymbol{E}M_{pi})/n_i}{\boldsymbol{E}M_{pi}},$$

which means that

$$n_i\zeta = \frac{\boldsymbol{E}M_p - \boldsymbol{E}M_{pi}}{\boldsymbol{E}M_{pi}}.$$

By contrast, for sampling from a finite universe, the signal noise ratio for a single item is

$$\frac{(\boldsymbol{E}M_p - c_i\boldsymbol{E}M_{pi})/n_i}{\boldsymbol{E}M_{pi}},$$

with the noise still being $\boldsymbol{E}M_{pi}$ for a single item. For sampling n_i items from a universe of size N_i, the noise becomes $c_i\boldsymbol{E}M_{pi}/n_i$, which leads to

$$\frac{(\boldsymbol{E}M_p - c_i\boldsymbol{E}M_{pi})}{c_i\boldsymbol{E}M_{pi}} = \frac{c_i}{n_i}\left(\zeta + \frac{1}{N_i}\right).$$

6.9 Using Equation 6.32, we want

$$\boldsymbol{E}\rho^2 = \frac{n'_i\zeta_L}{1 + n'_i\zeta_L} \geq .85,$$

where ζ_L is the lower limit in Equation 6.30. Solving for ζ_L gives

$$\zeta_L \geq \frac{.85}{n'_i(1 - .85)}.$$

From Equations 6.30 and 6.31,

$$\zeta_L = \frac{1}{n_i}\left(\frac{M_p}{M_{pi}F_{\alpha:\eta_p\eta_{pi}}} - 1\right).$$

Since we desire a one-sided 90% confidence interval, $\alpha = .1$. (Recall the Burdick and Graybill, 1992, convention that α is the area to the right of the percentile point.). It follows that

$$\zeta_L = \frac{1}{4}\left[\frac{24.5}{1.2333(2.27302)} - 1\right] = 1.9349.$$

The inequality

$$1.9349 \geq \frac{.85}{n_i'(1-.85)}$$

is satisfied for $n_i' = 2.9287$. Therefore, at least three items are required to be 90% confident that $\boldsymbol{E}\rho^2 \geq .85$.

Chapter 7

7.8 The expected value of the observed score variance is

$$\begin{aligned}\boldsymbol{E}S^2(p) &= \boldsymbol{E}\left(\frac{\sum_p \overline{X}_p^2 - n_p\overline{X}^2}{n_p - 1}\right)\\ &= \frac{\boldsymbol{E}\left(\sum_p \overline{X}_p^2\right) - n_p\boldsymbol{E}(\overline{X}^2)}{n_p - 1},\end{aligned} \qquad (1)$$

with $\overline{X}_p$ and $\overline{X}$ defined by Equation 7.44. We need expressions for the expected values of the two quadratic forms, $\sum_p \overline{X}_p^2$ and $\overline{X}^2$, in terms of variance components. Using a derivation similar to that used in Exercise 7.4 we obtain

$$\boldsymbol{E}\left(\sum_p \overline{X}_p^2\right) = n_p\mu^2 + n_p\sigma^2(p) + \frac{n_p\sigma^2(i)}{\ddot{n}_i} + \frac{n_p\sigma^2(pi)}{\ddot{n}_i}, \qquad (2)$$

and

$$\begin{aligned}\boldsymbol{E}(\overline{X}^2) &= \mu^2 + \frac{\sigma^2(p)}{n_p} + \frac{1}{n_p}\left[\frac{1}{\ddot{n}_i} + \frac{1}{n_p}\sum\sum_{p\neq p'}\frac{\tilde{n}_{pp'}}{\tilde{n}_p\tilde{n}_{p'}}\right]\sigma^2(i)\\ &\quad + \frac{\sigma^2(pi)}{n_p\ddot{n}_i}.\end{aligned} \qquad (3)$$

Replacing Equations 2 and 3 in Equation 1 gives Equation 7.45.

Deriving Equations 2 and 3 is tedious but relatively straightforward, except for the following term that arises in deriving Equation 3:

$$\begin{aligned}&\boldsymbol{E}\left[\sum_p\left(\frac{\sum_i \nu_i}{\tilde{n}_p}\right)\right]^2 =\\ &\quad\boldsymbol{E}\left[\sum_p\left(\frac{\sum_i \nu_i}{\tilde{n}_p}\right)^2 + \sum\sum_{p\neq p'}\left(\frac{\sum_i \nu_i}{\tilde{n}_p}\right)\left(\frac{\sum_i \nu_i}{\tilde{n}_{p'}}\right)\right].\end{aligned} \qquad (4)$$

The first term to the right of the equal sign is

$$\sum_p \left[\boldsymbol{E} \left(\frac{\sum_i \nu_i}{\widetilde{n}_p} \right)^2 \right] = \sum_p \frac{\sigma^2(i)}{\widetilde{n}_p} = \frac{n_p \sigma^2(i)}{\ddot{n}_i}.$$

The second term to the right of the equal sign is

$$\sum_{p \neq p'}\sum \left[\boldsymbol{E} \left(\frac{\sum_i \nu_i}{\widetilde{n}_p} \right) \left(\frac{\sum_i \nu_i}{\widetilde{n}_{p'}} \right) \right].$$

In thinking about this term, consider the special case of two persons and three items. Suppose that the first person answered the first two items, and the second person answered all three items. In this case, letting numerical subscripts of ν designate levels of i, the term becomes

$$\boldsymbol{E} \left(\frac{\nu_1 + \nu_2}{2} \right) \left(\frac{\nu_1 + \nu_2 + \nu_3}{3} \right) = \boldsymbol{E} \left(\frac{\nu_1\nu_1 + \nu_2\nu_2}{6} \right),$$

because $\boldsymbol{E}\nu_i\nu_j = 0$ for $i \neq j$. This simplifies to $2\sigma^2(i)/6$, where the '2' is the number of items responded to by both persons. Extending this logic to all pairs of persons, the second term to the right of the equal sign in Equation 4 is

$$\left[\sum_{p \neq p'}\sum \frac{\widetilde{n}_{pp'}}{\widetilde{n}_p \widetilde{n}_{p'}} \right] \sigma^2(i).$$

It follows that Equation 4 is

$$\boldsymbol{E} \left[\sum_p \left(\frac{\sum_i \nu_i}{\widetilde{n}_p} \right) \right]^2 = \left[\frac{n_p}{\ddot{n}_i} + \sum_{p \neq p'}\sum \frac{\widetilde{n}_{pp'}}{\widetilde{n}_p \widetilde{n}_{p'}} \right] \sigma^2(i).$$

7.9 A person's total observed score is $X_p = \sum_i X_{pi}$; the mean, over persons, of these total scores is $\overline{X}^+ = \sum_p X_p/n_p$; and the expected observed variance of the total scores is

$$\begin{aligned} \boldsymbol{E}S^2(X_p) &= \boldsymbol{E} \left[\frac{\sum_p X_p^2 - n_p(\overline{X}^+)^2}{n_p - 1} \right] \\ &= \frac{\boldsymbol{E}\left(\sum_p X_p^2\right) - n_p \boldsymbol{E}(\overline{X}^+)^2}{n_p - 1}. \end{aligned} \tag{5}$$

Therefore, we need to determine the expected value of the two quadratic forms in Equation 5. For the first one,

$$\boldsymbol{E}\left(\sum_p X_p^2\right) = \boldsymbol{E} \left[\sum_{p=1}^{n_p} \left(\sum_{i=1}^{n_{i:p}} X_{pi} \right)^2 \right]$$

$$
\begin{aligned}
&= \boldsymbol{E}\left[\sum_p \left(n_{i:p}\mu + n_{i:p}\nu_p + \sum_i \nu_{i:p}\right)^2\right] \\
&= \boldsymbol{E}\left\{\sum_p \left[n_{i:p}^2\mu^2 + n_{i:p}^2\nu_p^2 + \left(\sum_i \nu_{i:p}\right)^2\right]\right\} \\
&= \left(\textstyle\sum_p n_{i:p}^2\right)\mu^2 + \left(\textstyle\sum_p n_{i:p}^2\right)\sigma^2(p) + n_+\sigma^2(i{:}p).
\end{aligned} \tag{6}
$$

For the second quadratic form in Equation 5

$$
\begin{aligned}
\boldsymbol{E}(\overline{X}^+)^2 &= \boldsymbol{E}\left[\frac{1}{n_p}\sum_{p=1}^{n_p}\left(\sum_{i=1}^{n_{i:p}} X_{pi}\right)\right]^2 \\
&= \boldsymbol{E}\left[\frac{1}{n_p}\sum_p \left(n_{i:p}\mu + n_{i:p}\nu_p + \sum_i \nu_{i:p}\right)\right]^2 \\
&= \boldsymbol{E}\left[\frac{n_+}{n_p}\mu + \frac{1}{n_p}\sum_p n_{i:p}\nu_p + \frac{1}{n_p}\sum_p\sum_i \nu_{i:p}\right]^2 \\
&= \frac{n_+^2}{n_p^2}\mu^2 + \frac{\sum_p n_{i:p}^2}{n_p^2}\sigma^2(p) + \frac{n_+}{n_p^2}\sigma^2(i{:}p).
\end{aligned} \tag{7}
$$

Replacing Equations 6 and 7 in Equation 5 gives the expected observed score variance in Equation 7.52.

Chapter 8

8.4 Based on results reported in Section 8.2, for districts, $\widehat{S/N}(\Delta) = .00213/.00300 = .71$. Using the estimated variance components in Table 8.3 on page 5, the signal/noise ratio for persons in a randomly selected district is

$$
\widehat{S/N}(\Delta) = \frac{\hat\sigma^2(p{:}d)}{\dfrac{\hat\sigma^2(i)}{n_i} + \dfrac{\hat\sigma^2(pi{:}d)}{n_i}} = \frac{.03412}{\dfrac{.01509}{40} + \dfrac{.19490}{40}} = \frac{.03412}{.00525} = 6.50.
$$

To obtain the signal/noise ratio for pupils across all districts, we need estimates of the variance components for the $p \times i$ design based on pupils over all districts. These can be obtained from the estimates in Table 8.3 for the $(p{:}d) \times i$ design. An estimate of $\sigma^2(p)$ for the $p \times i$ design is simply

$$
\hat\sigma^2(d) + \hat\sigma^2(p{:}d) = .00213 + .03412 = .03625,
$$

an estimate of $\sigma^2(pi)$ for the $p \times i$ design is simply

$$\hat{\sigma}^2(di) + \hat{\sigma}^2(pi{:}d) = .00228 + .19490 = .19718,$$

and $\hat{\sigma}^2(i) = .01509$ is an estimate of $\sigma^2(i)$ for either design. It follows that the signal/noise ratio for pupils across all districts is

$$\widehat{S/N}(\Delta) = \frac{.03625}{\dfrac{.01509}{40} + \dfrac{.19718}{40}} = \frac{.03626}{.00531} = 6.83.$$

It is also possible to estimate the variance components for the across-all-districts $p \times i$ design using the T terms in Table 8.3 for the $(p{:}d) \times i$ design to obtain the T terms for the $p \times i$ design. Specifically, $T(p{:}d)$ is $T(p)$ for the $p \times i$ design, $T(pi{:}d)$ is $T(pi)$ for the $p \times i$ design, and $T(i)$ is the same for both designs. Using these equalities, variance components for the $p \times i$ design can be estimated in the usual manner. Doing so gives $\hat{\sigma}^2(p) = .03619$, $\hat{\sigma}^2(i) = .01515$, and $\hat{\sigma}^2(pi) = .19712$. These estimates are quite close to those obtained from the $(p{:}d) \times i$ design, but they pretend that persons are sampled from an undifferentiated population, which is not true.

8.5 The question asks for $\Pr(\overline{X}_p < 3.5 \mid \mu_p = 4)$. To answer the question, we need to determine the variance of the distribution of observed scores for students with a universe score of 4, which is the distribution of Δ-type errors for students with a universe score of 4. Ideally, this question would be answered using a conditional absolute SEM, but such information is not available to us. Therefore, we will use the overall absolute SEM, which is

$$\hat{\sigma}(\Delta) = \sqrt{\frac{.22713}{3} + \frac{.02300}{19} + \frac{.00245}{19} + \frac{.18968}{3(19)}} = .28351.$$

It follows that an average rating of 3.5 is $.5/.28351 = 1.76361$ z-scores to the left of a universe score of 4, which gives a probability of .039 for an average rating of 3.5 or lower.

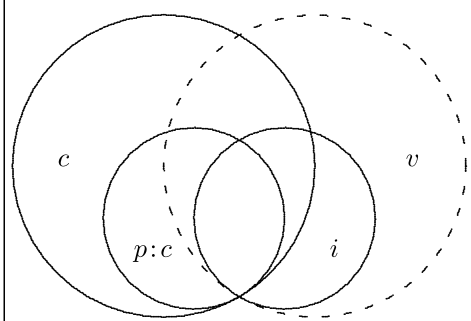
8.6 We need the variance of the distribution of $\overline{X}_p - \overline{X}_q$ for students with the same universe score. Since these students have taken the same instrument, this is the variance of the difference in their δ-type errors, which is $\sigma^2(\delta_p - \delta_q) = 2\sigma^2(\delta)$. The estimated SEM for the difference scores is

$$\hat{\sigma}(\delta_p - \delta_q) = \sqrt{2\left(\frac{.22713}{3} + \frac{.00245}{19} + \frac{.18968}{3(19)}\right)} = .39792,$$

which corresponds to a z-score of $.5/.39392 = 1.25654$. The probability that $|z| > 1.25654$ is .209.

Chapter 9

9.3



$$\boldsymbol{\Sigma}_c = \begin{bmatrix} \text{X} & \text{X} \\ \text{X} & \text{X} \end{bmatrix}$$

$$\boldsymbol{\Sigma}_{p:c} = \begin{bmatrix} \text{X} & \\ & \text{X} \end{bmatrix}$$

$$\boldsymbol{\Sigma}_i = \begin{bmatrix} \text{X} & \\ & \text{X} \end{bmatrix}$$

$$\boldsymbol{\Sigma}_{ci} = \begin{bmatrix} \text{X} & \\ & \text{X} \end{bmatrix}$$

$$\boldsymbol{\Sigma}_{pi:c} = \begin{bmatrix} \text{X} & \\ & \text{X} \end{bmatrix}$$

		v_1						v_2							
c	$p:c$	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8	i_9	i_{10}	i_{11}	i_{12}	i_{13}	i_{14}
1	1	X	X	X	X	X	X	X	X	X	X	X	X	X	X
1	2	X	X	X	X	X	X	X	X	X	X	X	X	X	X
1	3	X	X	X	X	X	X	X	X	X	X	X	X	X	X
2	1	X	X	X	X	X	X	X	X	X	X	X	X	X	X
2	2	X	X	X	X	X	X	X	X	X	X	X	X	X	X

9.6

$$\mathbf{M}_p = \begin{bmatrix} 3.4611 & 2.6204 \\ 2.6204 & 3.7500 \end{bmatrix} \qquad \widehat{\boldsymbol{\Sigma}}_p = \begin{bmatrix} .3107 & .3039 \\ .3039 & .3156 \end{bmatrix}$$

$$\mathbf{M}_{i:p} = \begin{bmatrix} 1.5967 & .7967 \\ .7967 & 1.8567 \end{bmatrix} \qquad \widehat{\boldsymbol{\Sigma}}_{i:p} = \begin{bmatrix} 1.5967 & .7967 \\ .7967 & 1.8567 \end{bmatrix}$$

Note that $\mathbf{M}_p$ is necessarily the same for both the $p^\bullet \times i^\bullet$ and $i^\bullet : p^\bullet$ designs. Given the computations provided in Table 9.3, the elements of $\mathbf{M}_{i:p}$ are obtained most easily by noting that the formula for $T(i{:}p)$ for the $i^\bullet : p^\bullet$ design is identical to that for $T(pi)$ for the $p^\bullet \times i^\bullet$ design. Similarly, the formula for $TP_{vv'}(i{:}p)$ for the $i^\bullet : p^\bullet$ design is identical to that for $TP_{vv'}(pi)$ for the $p^\bullet \times i^\bullet$ design.

9.8 Using Equation 6.2, $\hat{\sigma}[\hat{\sigma}^2_v(p)] = .250$ and $\hat{\sigma}[\hat{\sigma}^2_{v'}(p)] = .272$. Both of these are larger than $\hat{\sigma}[\hat{\sigma}_{vv'}(p)] = .227$ reported in Section 9.4.3 (immediately after Equation 9.54).

Chapter 10

10.3 The weights for the three categories and the category means are as follows:

v	w_v	a_v	$a_v - w_v$	$\overline{X}_v$
1	.25	0	−.25	2.5
2	.50	1	.50	4.5
3	.25	0	−.25	6.5

It follows that the first term in Equation 10.36 is zero. The second term is obtained by multiplying each element in $\widehat{\boldsymbol{\Sigma}}_p$ by the corresponding element in the following matrix of weights

$$\begin{bmatrix} .0625 & -.125 & .0625 \\ -.1250 & .250 & -.1250 \\ .0625 & -.125 & .0625 \end{bmatrix}$$

which gives a result of .6384. The third term is simply

$$\frac{\hat{\sigma}_2^2(i) + \hat{\sigma}_2^2(pi)}{8} = \frac{.1994 + 1.0714}{8} = .1589.$$

Consequently,

$$MSE_C(\Delta) = 0 + .6384 + .1589 = .797.$$

10.4 We need to determine a condition under which

$$\frac{\hat{\sigma}_{12}(pI)}{\hat{\sigma}_1^2(pI)} > \frac{\hat{\sigma}_{12}(I) + \hat{\sigma}_{12}(pI)}{\hat{\sigma}_1^2(I) + \hat{\sigma}_1^2(pI)}.$$

Using standard algebraic procedures, we find that $\hat{\rho}_{12}(\delta) > \hat{\rho}_{12}(\Delta)$ when

$$\frac{\hat{\sigma}_1^2(I)}{\hat{\sigma}_1^2(pI)} > \frac{\hat{\sigma}_{12}(I)}{\hat{\sigma}_{12}(pI)}$$

or, equivalently, when $\hat{\rho}_{12}(I) < \hat{\rho}_{12}(pI)$.

10.7 The basic structure of the design is $I^\bullet\!:\!p^\bullet$. Recall that v_1 is accuracy and v_2 is speed. The primary complexity is that n_{i1} is specified to be two, and we are asked to find n_{i2}. It is convenient to begin by noting from Table 9.3 that

$$\widehat{\boldsymbol{\Sigma}}_{i:p} = \widehat{\boldsymbol{\Sigma}}_i + \widehat{\boldsymbol{\Sigma}}_{pi} = \begin{bmatrix} 1.5966 & .7967 \\ .7967 & 1.8567 \end{bmatrix}.$$

It follows that

$$\widehat{\boldsymbol{\Sigma}}_\Delta = \begin{bmatrix} 1.5966/2 & .7967/2 \\ .7967/2 & 1.8567/n_{i2} \end{bmatrix}.$$

Note that the divisor of the covariance component is two because that is the number of items that are scored for *both* accuracy and speed. We need to find the value of n_{i2} such that

$$\hat{\sigma}_C(\Delta) = .5\sqrt{\frac{1.5966}{2} + 2\left(\frac{.7967}{2}\right) + \frac{1.8567}{n_{i2}}} \leq \frac{2}{3}.$$

This inequality is satisfied for $n_{i2} = 10.16$, which means that we must set $n_{i2} = 11$ to have $\hat{\sigma}_C(\Delta) \leq 2/3$.

10.9 For the first person's observed scores, the variances for the three levels of v are .5000, .9167, and 2.0000, respectively. Since the design is $p^\bullet \times I^\circ$, there are no covariance components. Therefore,

$$\begin{aligned}\hat{\sigma}_C(\Delta_1) &= \sqrt{(.25)^2\left(\frac{.5}{2}\right) + (.50)^2\left(\frac{.9167}{4}\right) + (.25)^2\left(\frac{2}{2}\right)} \\ &= .3680.\end{aligned}$$

10.10 The proof proceeds much like that used in deriving Equation 10.51. We begin by noting that

$$\Pr(\text{overlap}) = 1 - \Pr(\text{no overlap}) = 1 - \Pr(Y_H < X_L) + \Pr(X_H < Y_L).$$

Now,

$$\begin{aligned}\Pr(Y_H < X_L) &= \Pr\left[(Y + \sigma_E) - (X - \sigma_E) < 0\right] \\ &= \Pr(Y - X < -2\sigma_E) \\ &= \Pr(X - Y > 2\sigma_E) \\ &= 1 - \Pr(X - Y < 2\sigma_E),\end{aligned}$$

and

$$\begin{aligned}\Pr(X_H < Y_L) &= \Pr\left[(X + \sigma_E) - (Y - \sigma_E) < 0\right] \\ &= \Pr(X - Y < -2\sigma_E).\end{aligned}$$

Therefore,

$$\begin{aligned}\Pr(\text{overlap}) &= 1 - \Pr(Y_H < X_L) + \Pr(X_H < Y_L) \\ &= 1 - [1 - \Pr(X - Y < 2\sigma_E) + \Pr(X - Y < -2\sigma_E)] \\ &= \Pr(X - Y < 2\sigma_E) - \Pr(X - Y < -2\sigma_E).\end{aligned}$$

Under the normality assumptions,

$$(X - Y) \sim N(\tau_X - \tau_Y, 2\sigma_E^2),$$

the probability of overlap becomes

$$\begin{aligned}\Pr(\text{overlap}) &= \Pr\left(z < \frac{2\sigma_E - (\tau_X - \tau_Y)}{\sqrt{2\sigma_E^2}}\right) \\ &\quad - \Pr\left(z < \frac{-2\sigma_E - (\tau_X - \tau_Y)}{\sqrt{2\sigma_E^2}}\right).\end{aligned}$$

10.11 The two estimated conditional absolute standard errors are $\hat{\sigma}_1(\Delta_1) = \sqrt{.1778} = .4216$ and $\hat{\sigma}_2(\Delta_1) = \sqrt{.1333} = .3652$, and the estimated standard error of the observed difference score is

$$\hat{\sigma}_D(\Delta_1) = \sqrt{.1778 + .1333 - 2(.1000)} = .3333.$$

Using Equation 10.54

$$\begin{aligned}\Pr(\text{overlap}) &= \Pr\left(z < \frac{.4216 + .3652 - .5}{.3333}\right) \\ &\quad - \Pr\left(z < \frac{-(.4216 + .3652) - .5}{.3333}\right) \\ &= \Pr(z < .8604) - \Pr(z < -3.8604) \\ &= .805 - .000 \\ &= .805.\end{aligned}$$

10.13 A general equation for the estimated standard error of a D study covariance component is given by Equation 10.42. The special case given by Equation 10.45 can be used here, with t playing the role of i in Equation 10.45 and r playing the role of h, which leads to

$$\begin{aligned}\hat{\sigma}[\hat{\sigma}_{12}(\delta)] &= \left\{\left[\frac{1}{(25)(4)}\right]^2 \frac{(10.8626)(10.9400) + (7.4388)^2}{(59)(3) + 2}\right. \\ &\quad + \left[\frac{1}{(4)(2)}\right]^2 \frac{(1.8905)(2.2486) + (1.0794)^2}{(59)(24) + 2} \\ &\quad + \left[\frac{1}{(4)(2)} - \frac{1}{(25)(4)} - \frac{1}{(4)(2)}\right]^2 \\ &\quad \left.\frac{(1.3501)(1.4850) + (.6638)^2}{(59)(3)(24) + 2}\right\}^{1/2} \\ &= \sqrt{(.01)^2(.9730) + (.125)^2(.0038) + (.01)^2(.0006)} \\ &= .0125.\end{aligned}$$

10.14 Under normality assumptions, Equation 9.53 can be used to estimate the standard error, which is

$$\sqrt{\frac{MS_L(p)MS_W(p) + [MP_{LW}(p)]^2}{n_t^2[(n_p - 1) + 2]} + \frac{MS_L(pt)MS_W(pt) + [MP_{LW}(pt)]^2}{n_t^2[(n_p - 1)(n_t - 1) + 2]}}.$$

The mean squares and mean products can be obtained from the estimated variance and covariance components in the following manner:

$$
\begin{aligned}
MS_L(p) &= .307 + 12(.008) + 3(.398) + 36(.321) = 13.153 \\
MS_W(p) &= .249 + 12(.042) + 3(.151) + 36(.740) = 27.846 \\
MP_{LW}(p) &= .052 + 12(.403) = 4.888 \\
MS_L(pt) &= .307 + 3(.398) = 1.501 \\
MS_W(pt) &= .249 + 3(.151) = .702 \\
MP_{LW}(pt) &= .052.
\end{aligned}
$$

Replacing these values in the formula for the standard error gives

$$\sqrt{\frac{(13.153)(27.846) + (4.888)^2}{144(49+2)} + \frac{(1.501)(.702) + (.052)^2}{144[(49)(11)+2]}} = .231.$$

This estimate is based on normality assumptions, whereas the estimate of .074 in Table 10.7 is not. Also, .231 is an estimate of the standard error of $\sigma_{LW}(p)$ for a *single* form based on $n_t = 12$ and $n_r = 3$, whereas the estimated standard error of .074 is for the average of three estimates of $\hat{\sigma}_{LW}(p)$. Therefore, the normality-based estimate of .231 can be compared to the empirical estimate of $.074\sqrt{3} = .128$. The empirical estimate is nearly twice as large as the normality-based estimate. Unless there is strong reason to believe the normality assumptions, this casts serious doubt on the single-form estimate of .231.

Chapter 11

11.2 The equivalences are easily established by noting that for a balanced $i^\bullet\!:\!p^\bullet$ design, $n_{i:p} = r_i = \ddot{n}_i = n_i$ and

$$
\begin{aligned}
TP_{vv'}(p) &= n_i\, CP_{vv'}(p) \\
TP_{vv'}(i\!:\!p) &= CP_{vv'}(i\!:\!p) \\
TP_{vv'}(\mu) &= n_p n_i\, CP_{vv'}(\mu).
\end{aligned}
$$

11.4 Begin by noting that

$$\boldsymbol{E}\, CP_{vv'}(p) = \boldsymbol{E}\left(\sum_p \overline{X}_{pv}\overline{X}_{pv'}\right) = \sum_p \boldsymbol{E}\left(\overline{X}_{pv}\overline{X}_{pv'}\right).$$

Letting $\sum_h n_{i:h} = n_{i+}$ and $\sum_h m_{i:h} = m_{i+}$,

$$
\begin{aligned}
\overline{X}_{pv} &= \mu_v + \nu_p + \frac{\sum_h n_{i:h}\nu_h}{n_{i+}} + \frac{\sum_h \sum_i \nu_{i:h}}{n_{i+}} \\
&\quad + \frac{\sum_h n_{i:h}\nu_{ph}}{n_{i+}} + \frac{\sum_h \sum_i \nu_{pi:h}}{n_{i+}},
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{X}_{pv'} &= \mu_{v'} + \xi_p + \frac{\sum_h m_{i:h}\xi_h}{m_{i+}} + \frac{\sum_h \sum_i \xi_{i:h}}{m_{i+}} \\
&\quad + \frac{\sum_h m_{i:h}\xi_{ph}}{m_{i+}} + \frac{\sum_h \sum_i \xi_{pi:h}}{m_{i+}}.
\end{aligned}
$$

Because of the zero expectations given by Equations 9.29–9.31, it follows that $\boldsymbol{E}CP_{vv'}(p)$ has only four terms. The first term is

$$
\sum_p \boldsymbol{E}\mu_v\mu_{v'} = n_p\mu_v\mu_{v'},
$$

and the second term is

$$
\sum_p \boldsymbol{E}\nu_p\xi_p = n_p\,\sigma_{vv'}(p).
$$

The third term is

$$
\begin{aligned}
\sum_p \boldsymbol{E}\left[\left(\frac{\sum_h n_{i:h}\nu_h}{n_{i+}}\right)\left(\frac{\sum_h m_{i:h}\xi_h}{m_{i+}}\right)\right] &= \sum_p \left[\frac{\sum_h n_{i:h}m_{i:h}\boldsymbol{E}(\nu_h\xi_h)}{n_{i+}m_{i+}}\right] \\
&= n_p\left(\frac{\sum_h n_{i:h}m_{i:h}}{n_{i+}m_{i+}}\right)\sigma_{vv'}(h) \\
&= n_p t\,\sigma_{vv'}(h).
\end{aligned}
$$

The derivation of this term rather clearly demonstrates the rule in Equation 11.19 for obtaining the coefficient of a covariance component in an expected sum-of-cross-products term. A similar derivation for the fourth term gives

$$
\sum_p \boldsymbol{E}\left[\left(\frac{\sum_h n_{i:h}\nu_{ph}}{n_{i+}}\right)\left(\frac{\sum_h m_{i:h}\xi_{ph}}{m_{i+}}\right)\right] = n_p t\,\sigma_{vv'}(ph).
$$

11.6 For the $p^\bullet \times (i^\circ\!:\!h^\circ)$ design, prove that the expected value of the compound-means covariance in Equation 11.39 is $\sigma_{vv'}(p)$.

Answer. Given the definition of the compound mean in Equation 11.38, $\overline{X}^*_{pv}$ in terms of ν effects is

$$
\begin{aligned}
\overline{X}^*_{pv} &= \mu_v + \nu_p + \frac{\sum_h \nu_h}{n_h} + \frac{1}{n_h}\sum_h\left(\frac{\sum_i \nu_{i:h}}{n_{i:h}}\right) \\
&\quad + \frac{\sum_h \nu_{ph}}{n_h} + \frac{1}{n_h}\sum_h\left(\frac{\sum_i \nu_{pi:h}}{n_{i:h}}\right), \qquad (8)
\end{aligned}
$$

and there is a corresponding equation for $\overline{X}^*_{pv'}$ in terms of ξ effects. Since there are different levels of h and different levels of i in the

$p^\bullet \times (i^\circ : h^\circ)$ design, all covariance components involving h and/or i are zero. It follows that

$$\boldsymbol{E}\sum_p \overline{X}^*_{pv}\overline{X}^*_{pv'} = \sum_p \boldsymbol{E}(\mu_v + \nu_p)(\mu_v + \xi_p) = n_p\mu_v\mu_{v'} + n_p\sigma_{vv'}(p).$$

Similarly, $\overline{X}^*_v$ in terms of ν score effects is

$$\begin{aligned}\overline{X}^*_v &= \mu_v + \frac{\sum_p \nu_p}{n_p} + \frac{\sum_h \nu_h}{n_h} + \frac{1}{n_h}\sum_h\left(\frac{\sum_i \nu_{i:h}}{n_{i:h}}\right) \\ &\quad + \frac{\sum_p\sum_h \nu_{ph}}{n_p n_h} + \frac{1}{n_p}\sum_p\left[\frac{1}{n_h}\sum_h\left(\frac{\sum_i \nu_{pi:h}}{n_{i:h}}\right)\right], \qquad (9)\end{aligned}$$

and there is a corresponding equation for $\overline{X}^*_{v'}$ in terms of ξ score effects. Since all covariance components involving h and/or i are zero,

$$\boldsymbol{E}\overline{X}^*_v\overline{X}^*_{v'} = \boldsymbol{E}\left(\mu_v + \frac{\sum_p \nu_p}{n_p}\right)\left(\mu_v + \frac{\sum_p \xi_p}{n_p}\right) = \mu_v\mu_{v'} + \frac{\sigma_{vv'}(p)}{n_p}.$$

It follows that

$$\begin{aligned}\boldsymbol{E}S_{vv'}(\overline{X}^*_p) &= \frac{n_p}{n_p - 1}\left[\frac{n_p\mu_v\mu_{v'} + n_p\sigma_{vv'}(p)}{n_p} - \mu_v\mu_{v'} + \frac{\sigma_{vv'}(p)}{n_p}\right] \\ &= \sigma_{vv'}(p).\end{aligned}$$

11.7 We need to prove that $\boldsymbol{E}MP_{vv'}(h) = \sigma_{vv'}(ph) + n_h\sigma_{vv'}(p)$ where

$$MP_{vv'}(p) = \frac{TP_{vv'}(p) - TP_{vv'}(\mu)}{n_p - 1},$$

$TP_{vv'}(p) = n_h\sum_p \overline{X}^*_{pv}\overline{X}^*_{pv'}$, $TP_{vv'}(\mu) = n_p n_h \overline{X}^*_v\overline{X}^*_{v'}$, $\overline{X}^*_{pv}$ is given by Equation 8, and $\overline{X}^*_v$ is given by Equation 9.

Since the levels of i are different for v and v', all covariance components involving i are zero. It follows that

$$\begin{aligned}\boldsymbol{E}TP_{vv'}(p) &= \boldsymbol{E}\left(n_h\sum_p \overline{X}^*_{pv}\overline{X}^*_{pv'}\right) \\ &= n_h\sum_p \boldsymbol{E}(\overline{X}^*_{pv}\overline{X}^*_{pv'}) \\ &= n_h\sum_p \boldsymbol{E}\Bigg[\mu_v\mu_{v'} + \nu_p\xi_p + \left(\frac{\sum_h \nu_h}{n_h}\right)\left(\frac{\sum_h \xi_h}{n_h}\right) \\ &\quad + \left(\frac{\sum_h \nu_{ph}}{n_h}\right)\left(\frac{\sum_h \xi_{ph}}{n_h}\right)\Bigg] \\ &= n_p n_h\mu_v\mu_{v'} + n_p n_h\sigma_{vv'}(p) + n_p\sigma_{vv'}(h) + n_p\sigma_{vv'}(ph).\end{aligned}$$

A similar derivation gives

$$\begin{aligned}
\boldsymbol{E}\,TP_{vv'}(\mu) &= \boldsymbol{E}\left(n_p n_h \overline{X}^*_v \overline{X}^*_{v'}\right) \\
&= n_h n_h \boldsymbol{E}(\overline{X}^*_v \overline{X}^*_{v'}) \\
&= n_p n_h \mu_v \mu_{v'} + n_h \sigma_{vv'}(p) + n_p \sigma_{vv'}(h) + \sigma_{vv'}(ph).
\end{aligned}$$

It follows that

$$\begin{aligned}
\boldsymbol{E}MP_{vv'}(p) &= \frac{[n_p n_h \sigma_{vv'}(p) - n_p \sigma_{vv'}(p)] + [n_p \sigma_{vv'}(ph) - \sigma_{vv'}(ph)]}{n_p - 1} \\
&= \sigma_{vv'}(ph) + n_h \sigma_{vv'}(p).
\end{aligned}$$

11.9 For any given p, the number of responses for v is $\widetilde{n}_p$, and the number of responses for v' is $\widetilde{m}_p$. It follows that

$$\overline{X}_{pv} = \mu_v + \nu_p + \sum_i \frac{\nu_i}{\widetilde{n}_p} + \sum_i \frac{\nu_{pi}}{\widetilde{n}_p}$$

and

$$\overline{X}_{pv'} = \mu_{v'} + \xi_p + \sum_i \frac{\xi_i}{\widetilde{m}_p} + \sum_i \frac{\xi_{pi}}{\widetilde{m}_p}.$$

Therefore,

$$\begin{aligned}
\boldsymbol{E}\left(\sum_p \overline{X}_{pv} \overline{X}_{pv'}\right) &= n_p \mu_v \mu_{v'} + n_p \sigma_{vv'}(p) \\
&\quad + \boldsymbol{E}\left[\sum_p \left(\frac{\sum_i \nu_i}{\widetilde{n}_p}\right)\left(\frac{\sum_i \xi_i}{\widetilde{m}_p}\right)\right] \\
&\quad + \boldsymbol{E}\left[\sum_p \left(\frac{\sum_i \nu_{pi}}{\widetilde{n}_p}\right)\left(\frac{\sum_i \xi_{pi}}{\widetilde{m}_p}\right)\right].
\end{aligned}$$

For each of the last two terms, the expected value of the product of the ν and ξ effects is non-zero only when the level of i is the same for both v and v'. Let q_p designate the number of items for person p such that neither the response for v nor the response for v' is missing. Then

$$\begin{aligned}
\boldsymbol{E}\left(\sum_p \overline{X}_{pv} \overline{X}_{pv'}\right) &= n_p \mu_v \mu_{v'} + n_p \sigma_{vv'}(p) + \sum_p \frac{q_p}{\widetilde{n}_p \widetilde{m}_p} \sigma_{vv'}(i) \\
&\quad + \sum_p \frac{q_p}{\widetilde{n}_p \widetilde{m}_p} \sigma_{vv'}(pi).
\end{aligned}$$

11.12 For this design, conditional absolute error variance is the variance of the mean for the within-person $I^\circ\!:\!H^\bullet$ design, in the sense discussed in Section 10.2.3. First, we need to obtain

$$\widehat{\boldsymbol{\Sigma}}_h = \begin{bmatrix} \hat\sigma_1^2(h) & \hat\sigma_{12}(h) \\ \hat\sigma_{12}(h) & \hat\sigma_2^2(h) \end{bmatrix} \quad \text{and} \quad \widehat{\boldsymbol{\Sigma}}_{i:h} = \begin{bmatrix} \hat\sigma_1^2(i{:}h) & \\ & \hat\sigma_2^2(i{:}h) \end{bmatrix}.$$

Then, the estimate of conditional absolute error variance is

$$\hat\sigma_C^2(\Delta_p) = \left[w_1^2\,\frac{\hat\sigma_1^2(h)}{\breve n'_{h_1}} + w_2^2\,\frac{\hat\sigma_2^2(h)}{\breve n'_{h_2}} + 2w_1w_2\,\frac{\hat\sigma_{12}(h)}{\breve n'_{h_{12}}}\right] + \left[w_1^2\,\frac{\hat\sigma_1^2(i{:}h)}{n'_{i+}} + w_2^2\,\frac{\hat\sigma_2^2(i{:}h)}{m'_{i+}}\right],$$

where $w_1 = .523$, $w_2 = .477$, $\breve n'_{h_1} = 4.372$, $\breve n'_{h_2} = 4.546$, $\breve n'_{h_{12}} = 5.616$, $n'_{i+} = 23$, and $m'_{i+} = 21$, as shown in Section 11.2.2.

The estimated covariance component in $\widehat{\boldsymbol{\Sigma}}_h$ is the observed covariance of the h mean scores. It is straightforward to determine that $\hat\sigma_{12}(h) = .0045$. The estimated variance components can be obtained using the formulas in Section 7.1.2, because the unbalanced $i\!:\!h$ design is formally identical to the unbalanced $i\!:\!p$ design. Recalling that these are dichotomous data, the analogous T terms are given in the following table:

	$T(h)$	$T(i{:}h)$	$T(\mu)$	n_+	r_i
v_1	17.9167	20	17.3913	23	5.2609
v_2	10.6667	14	9.3333	21	4.6191

Equations 7.9 and 7.10 (with the obvious replacement of p with h) can be used to obtain the G study estimated variance components. It follows that

$$\widehat{\boldsymbol{\Sigma}}_h = \begin{bmatrix} .0035 & .0045 \\ .0045 & .0305 \end{bmatrix} \quad \text{and} \quad \widehat{\boldsymbol{\Sigma}}_{i:h} = \begin{bmatrix} .1157 & \\ & .2083 \end{bmatrix},$$

and

$$\begin{aligned}\hat\sigma_C^2(\Delta_p) &= \left[(.523)^2\,\frac{.0035}{4.372} + (.477)^2\,\frac{.0305}{4.546} + 2(.477)(.523)\,\frac{.0045}{5.616}\right] \\ &\quad + \left[(.523)^2\,\frac{.1157}{23} + (.477)^2\,\frac{.2083}{21}\right] \\ &= .0058.\end{aligned}$$

Therefore, $\hat\sigma_C(\Delta_p) = .076$. Since this person's composite score is

$$\overline{X}_{pC} = .523(20/23) + .477(14/21) = .773,$$

the fitted value is

$$\hat\sigma_C(\Delta_p) = \sqrt{-.0038 + .0537\,(.773) - .0529\,(.773)^2} = .078.$$

Chapter 12

12.2 In the derivation of Equation 12.23, $S_{T_1X_2}$ equals the covariance between true scores, whereas under classical test theory assumptions, $S_{T_1X_2}$ equals the covariance between observed scores (or, more explicitly, the covariance between true scores equals the covariance between observed scores) as indicated in the following derivation:

$$r_{T_1X_2} = \frac{S_{T_1X_2}}{S_{T_1}S_{X_2}} = \frac{S_{X_1X_2}}{S_{T_1}S_{X_2}} = \frac{S_{X_1}r_{X_1X_2}}{S_{T_1}} = \frac{r_{X_1X_2}}{\sqrt{\boldsymbol{E}\rho_1^2}}.$$

12.4 Since a regression equation is simply a weighted sum of observed variables, it necessarily follows that

$$\sigma_v^2(\hat{\mu}_p) = \sum_{j=1}^{n_v}\sum_{j'=1}^{n_v} b_{jv}b_{j'v}S_{jj'} = \sum_{j=1}^{n_v} b_{jv}\left(\sum_{j'=1}^{n_v} b_{j'v}S_{jj'}\right).$$

Using the result derived in Exercise 12.5, the term in parentheses is

$$\sum_{j'=1}^{n_v} b_{j'v}S_{jj'} = \sigma_{vj},$$

or, interchanging the roles of j and j',

$$\sum_{j=1}^{n_v} b_{jv}S_{jj'} = \sigma_{vj'}.$$

Now, the covariance between the regressed score estimates for v and v' is, in general,

$$\sigma_{vv'}(\hat{\mu}_p) = \sum_{j=1}^{n_v}\sum_{j'=1}^{n_v} b_{jv}b_{j'v'}S_{jj'} = \sum_{j'=1}^{n_v} b_{j'v'}\left(\sum_{j=1}^{n_v} b_{jv}S_{jj'}\right).$$

Replacing the quantity in parentheses with the previously derived result,

$$\sigma_{vv'}(\hat{\mu}_p) = \sum_{j'=1}^{n_v} b_{j'v'}\sigma_{vj'},$$

or, equivalently,

$$\sigma_{vv'}(\hat{\mu}_p) = \sum_{j=1}^{n_v} b_{jv'}\sigma_{vj}.$$

12.6 Using the standard score formulas in Table 12.2,

$$\begin{aligned} R_{12} &= \frac{\sigma_{12}(\hat{Z}_p)}{\rho_{12}} \\ &= \frac{\beta_{11}\rho_1\rho_{21} + \beta_{21}\rho_2\rho_{22}}{\rho_{12}} \\ &= \frac{(.6684)(\sqrt{.6382})(.8663) + (.1794)(\sqrt{.5902})(1)}{.8663} \\ &= .693. \end{aligned}$$

Alternatively, the covariance of the predicted standard scores (numerator of this result) can be obtained through direct computation using the scores in Table 12.1. It is simply the average of the cross products of the predicted standard scores.

12.8 (a) Normal equations for v_1 in the $I^\bullet\!:\!p^\bullet$ design with $n_i' = 6$:

$$\begin{aligned} .6342b_1 + .4520b_2 &= .3682 \\ .4520b_1 + .6783b_2 &= .3193. \end{aligned}$$

(b) Normal equations for v_2 in the $I^\bullet\!:\!p^\bullet$ design with $n_i' = 8$:

$$\begin{aligned} .5677b_1 + .4189b_2 &= .3193 \\ .4189b_1 + .6010b_2 &= .3689. \end{aligned}$$

(c) Normal equations for v_1 in the $I^\circ\!:\!p^\bullet$ design with $n_i' = 6$:

$$\begin{aligned} .6343b_1 + .3193b_2 &= .3682 \\ .3193b_1 + .6783b_2 &= .3193. \end{aligned}$$

(d) Normal equations for v_2 in the $I^\circ\!:\!p^\bullet$ design with $n_i' = 8$:

$$\begin{aligned} .5677b_1 + .3193b_2 &= .3193 \\ .3193b_1 + .6010b_2 &= .3689. \end{aligned}$$

12.10 When $\rho_1 = \rho_2$ and $S_1 = S_2$, it necessarily follows that $\sigma_1 = \sigma_2$. We will designate these common values as ρ, S, and σ, respectively. For the two-variable prediction, under these simplifying assumptions, Equation 12.82 gives

$$R_C^2 = \frac{b_2 - b_1}{2},$$

where b_1 and b_2 can be obtained using Equations 12.79 and 12.80. Equation 12.79 gives

$$b_1 = \frac{1}{1 - r_{12}^2}\left[\frac{\sigma_{12}}{S^2} - \rho^2 - r_{12}\left(\frac{\sigma^2 - \sigma_{12}}{S^2}\right)\right]$$

$$
\begin{aligned}
&= \frac{1}{1-r_{12}^2}\left[\frac{\sigma_{12}}{S^2} - \frac{\sigma^2}{S^2} - r_{12}\left(\frac{\sigma^2-\sigma_{12}}{S^2}\right)\right] \\
&= \frac{1}{1-r_{12}^2}\left[\frac{\sigma_{12}-\sigma^2}{S^2} - r_{12}\left(\frac{\sigma^2-\sigma_{12}}{S^2}\right)\right] \\
&= -\left(\frac{1+r_{12}}{1-r_{12}^2}\right)\left(\frac{\sigma^2-\sigma_{12}}{S^2}\right) \\
&= -\left(\frac{1}{1-r_{12}}\right)\left(\frac{\sigma^2-\sigma_{12}}{S^2}\right).
\end{aligned}
$$

Similarly, Equation 12.80 gives

$$b_2 = \left(\frac{1}{1-r_{12}}\right)\left(\frac{\sigma^2-\sigma_{12}}{S^2}\right).$$

It follows that

$$
\begin{aligned}
R_C^2 &= \frac{1}{2}\left(\frac{2}{1-r_{12}}\right)\left(\frac{\sigma^2-\sigma_{12}}{S^2}\right) \\
&= \frac{S^2}{S^2-S_{12}}\left(\frac{\sigma^2-\sigma_{12}}{S^2}\right) \\
&= \frac{\sigma^2-\sigma_{12}}{S^2-S_{12}},
\end{aligned}
$$

which is identical to the result obtained from Equation 12.86 for the single-variable prediction. (Recall that $\boldsymbol{E}\rho^2 = R^2$ for the single-variable prediction.)

12.13 Let X_{p1} and X'_{p1} be observed scores for persons on randomly parallel tests for v_1, which we abbreviate X_1 and X'_1, respectively. The variables X_2 and X'_2 are defined similarly. As is the convention in generalizability theory, we will assume these raw scores are in the mean score metric. Predicted, randomly parallel composites are

$$
\begin{aligned}
\hat{Y} &= b_0 + b_1X_1 + b_2X_2 \\
\hat{Y}' &= b_0 + b_1X'_1 + b_2X'_2
\end{aligned}
$$

with covariance

$$\sigma_{\hat{Y}\hat{Y}'} = b_1^2 S^2_{X_1X'_1} + b_2^2 S^2_{X_2X'_2} + b_1b_2S_{X_1X'_2} + b_1b_2S_{X'_1X_2}.$$

Now

$$S^2_{X_1X'_1} = S_{(T_1+E_1)(T'_1+E'_1)} = S_{T_1T'_1} + S_{E_1E'_1} = \sigma_1^2$$

because errors are uncorrelated for different instances of a measurement procedures. Similarly,

$$S^2_{X_2X'_2} = \sigma_2^2.$$

The covariance of X_1 with X_2' is

$$S_{X_1X_2'} = S_{(T_1+E_1)(T_2'+E_2')} = S_{T_1T_2'} + S_{E_1E_2'} = \sigma_{12}.$$

Similarly,

$$S_{X_1'X_2} = \sigma_{12}.$$

Note that, in these derivations, $S_{E_1E_2'} = S_{E_1'E_2} = 0$ even if $S_{E_1E_2}$ is non-zero. The fact that errors are correlated for a *particular* instance of the measurement does not mean they are correlated for different instances (i.e., different sets of items). It follows that

$$\sigma_{\hat{Y}\hat{Y}'} = b_1^2\sigma_1^2 + b_2^2\sigma_2^2 + 2b_1b_2\sigma_{12}.$$

For either $\hat{Y}$ and $\hat{Y}'$ the variance-of-a-sum formula gives

$$\sigma_{\hat{Y}}^2 = \sigma_{\hat{Y}'}^2 = b_1^2S_1^2 + b_2^2S_2^2 + 2b_1b_2S_{12}.$$

The ratio of the covariance to the variance is

$$\rho_{\hat{Y}\hat{Y}'} = \frac{\sigma_{\hat{Y}\hat{Y}'}}{\sigma_{\hat{Y}}^2} = \frac{b_1^2\sigma_1^2 + b_2^2\sigma_2^2 + 2b_1b_2\sigma_{12}}{b_1^2S_1^2 + b_2^2S_2^2 + 2b_1b_2S_{12}},$$

which is Equation 12.96. It can also be expressed as

$$\begin{aligned}\rho_{\hat{Y}\hat{Y}'} &= 1 - \frac{b_1^2(S_1^2 - \sigma_1^2) + b_2^2(S_2^2 - \sigma_2^2) + 2b_1b_2(S_{12} - \sigma_{12})}{b_1^2S_1^2 + b_2^2S_2^2 + 2b_1b_2S_{12}} \\ &= 1 - \frac{b_1^2\sigma_1^2(\delta) + b_2^2\sigma_1^2(\delta) + 2b_1b_2\sigma_{12}(\delta)}{b_1^2S_1^2 + b_2^2S_2^2 + 2b_1b_2S_{12}},\end{aligned}$$

which is Equation 12.97. Equation 12.71 shows that the denominator, $\sigma_{\hat{Y}}^2$, can also be expressed as

$$\sigma_{\hat{Y}}^2 = b_1(w_1\sigma_1^2 + w_2\sigma_{12}) + b_2(w_1\sigma_{12} + w_2\sigma_2^2),$$

which leads to Equation 12.98 for $\rho_{\hat{Y}\hat{Y}'}$.