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Notes about Partial Derivatives for Analytic Standard Errors of Levine-observed Equating with an External Anchor

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Abstract

Hanson, Zeng, and Kolen (1993) used the delta method to derive analytic standard errors of the Levine observed score and Levine true score equating methods with and without a normality assumption. They also provide the partial derivatives of the linear equating function $l(x)$ with respect to the 10 moments - $\mu_1(X)$, $\mu_1(V)$, $\sigma_1^2(X)$, $\sigma_1^2(V)$, $\sigma_1(X,V)$, $\mu_2(Y)$, $\mu_2(V)$, $\sigma_2^2(Y)$, $\sigma_2^2(V)$, and $\sigma_2(Y,V)$. In this note, it is shown that the partial derivative of $l(x)$ with respect to $\sigma_1^2(X)$ derived by Hanson, et al. (1993) has two places where signs are incorrect.

For the levine-observed equating method, the linear function estimated for equating X to the scale of Y is

$$l(x) = \frac{\sigma_s(Y)}{\sigma_s(X)}(x - \mu_s(X)) + \mu_s(Y),$$

where the subscript s refers to the synthetic population. Hanson, Zeng, and Kolen (1993)as well as Brennan and Kolen (2014) report that:

$$\begin{aligned}
\mu_s(X) &= \mu_1(X) - \omega_2\gamma_1[\mu_1(V) - \mu_2(V)],\\
\mu_s(Y) &= \mu_2(Y) + \omega_1\gamma_2[\mu_1(V) - \mu_2(V)],\\
\sigma_s^2(X) &= \sigma_1^2(X) - \omega_2\gamma_1^2[\sigma_1^2(V) - \sigma_2^2(V)] + \omega_1\omega_2\gamma_1^2[\mu_1(V) - \mu_2(V)]^2,\\
\sigma_s^2(Y) &= \sigma_2^2(Y) + \omega_1\gamma_2^2[\sigma_1^2(V) - \sigma_2^2(V)] + \omega_1\omega_2\gamma_2^2[\mu_1(V) - \mu_2(V)]^2,
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1 &= \frac{\sigma_1^2(X) + \sigma_1(X,V)}{\sigma_1^2(V) + \sigma_1(X,V)}, \text{ and}\\
\gamma_2 &= \frac{\sigma_2^2(Y) + \sigma_2(Y,V)}{\sigma_2^2(V) + \sigma_2(Y,V)}
\end{aligned}$$

for an external set of common items.

In Hanson, Zeng, and Kolen (1993), the derived partial derivative of $l(x)$ with respect to $\sigma_1^2(X)$ is as follows:

$$\begin{aligned}
&\frac{\partial l(x)}{\partial \sigma_1^2(X)}\\
&= \frac{Z_x\sigma_s(Y)}{\sigma_s^2(X)}\left\{\frac{1}{2} + \frac{\omega_2[\sigma_1^2(X) + \sigma_1(X,V)]\left[\sigma_2^2(V) - \sigma_1^2(V) + \omega_1[\mu_1(V) - \mu_2(V)]^2\right]}{[\sigma_1^2(V) + \sigma_1(X,V)]^2}\right\}\\
&\quad + \frac{\omega_2\sigma_s(Y)[\mu_1(V) - \mu_2(V)]}{\sigma_s(X)[\sigma_1^2(V) + \sigma_1(X,V)]}. \qquad (1)
\end{aligned}$$

where $Z_x = (x - \mu_s(X))/\sigma_s(X)$. This technical note finds that the term "$\frac{1}{2}+$" should be "$-\frac{1}{2}-$". Note that, with the multiplier of -1 in front of the large {} brackets, Equation 1 is equivalent to:

$$\begin{aligned}
&\frac{\partial l(x)}{\partial \sigma_1^2(X)}\\
&= (-1)\frac{Z_x\sigma_s(Y)}{\sigma_s^2(X)}\left\{-\frac{1}{2} - \frac{\omega_2[\sigma_1^2(X) + \sigma_1(X,V)]\left[\sigma_2^2(V) - \sigma_1^2(V) + \omega_1[\mu_1(V) - \mu_2(V)]^2\right]}{[\sigma_1^2(V) + \sigma_1(X,V)]^2}\right\}\\
&\quad + \frac{\omega_2\sigma_s(Y)[\mu_1(V) - \mu_2(V)]}{\sigma_s(X)[\sigma_1^2(V) + \sigma_1(X,V)]}. \qquad (2)
\end{aligned}$$

1 Partial Derivatives of γ_1, $\mu_s(X)$, and $\sigma_s^2(X)$ with respect to $\sigma_1^2(X)$

In order to obtain a partial derivative of $l(x)$ with respect to $\sigma_1^2(X)$, it is simplest to first obtain partial derivatives of γ_1, $\mu_s(X)$, and $\sigma_s^2(X)$ with respect to $\sigma_1^2(X)$.

1.1 Partial Derivative of γ_1 with respect to $\sigma_1^2(X)$

Since $\gamma_1 = \frac{\sigma_1^2(X)+\sigma_1(X,V)}{\sigma_1^2(V)+\sigma_1(X,V)}$,

$$\frac{\partial\gamma_1}{\partial\sigma_1^2(X)} = \frac{1}{\sigma_1^2(V)+\sigma_1(X,V)}.$$

1.2 Partial Derivative of $\mu_s(X)$ with respect to $\sigma_1^2(X)$

Since $\mu_s(X) = \mu_1(X) - \omega_2\gamma_1[\mu_1(V) - \mu_2(V)]$,

$$\begin{aligned}\frac{\partial\mu_s(X)}{\partial\sigma_1^2(X)} &= \frac{\partial\mu_s(X)}{\partial\gamma_1} \times \frac{\partial\gamma_1}{\partial\sigma_1^2(X)} \\ &= -\omega_2[\mu_1(V) - \mu_2(V)] \times \frac{1}{[\sigma_1^2(V)+\sigma_1(X,V)]} \\ &= \frac{-\omega_2[\mu_1(V) - \mu_2(V)]}{[\sigma_1^2(V)+\sigma_1(X,V)]}. \end{aligned} \tag{3}$$

1.3 Partial Derivative of $\sigma_s^2(X)$ with respect to $\sigma_1^2(X)$

Since

$$\begin{aligned}\sigma_s^2(X) &= \sigma_1^2(X) - \omega_2\gamma_1^2[\sigma_1^2(V) - \sigma_2^2(V)] + \omega_1\omega_2\gamma_1^2[\mu_1(V) - \mu_2(V)]^2 \\ &= \sigma_1^2(X) + \omega_2\gamma_1^2\left[\sigma_2^2(V) - \sigma_1^2(V) + \omega_1[\mu_1(V) - \mu_2(V)]^2\right], \end{aligned} \tag{4}$$

Suppose that the second term on the right-hand side in Equation 4 is denoted as a function h. The partial derivative of $\sigma_s^2(X)$ with respect to $\sigma_1^2(X)$, then, can be derived as follows:

$$\begin{aligned}\frac{\partial\sigma_s^2(X)}{\partial\sigma_1^2(X)} &= 1 + \frac{\partial h}{\partial\gamma_1} \times \frac{\partial\gamma_1}{\partial\sigma_1^2(X)} \\ &= 1 + \omega_2\left[\sigma_2^2(V) - \sigma_1^2(V) + \omega_1[\mu_1(V) - \mu_2(V)]^2\right] \times 2\gamma_1 \times \frac{1}{[\sigma_1^2(V)+\sigma_1(X,V)]} \\ &= 1 + \frac{2\omega_2[\sigma_1^2(X)+\sigma_1(X,V)]\left[\sigma_2^2(V) - \sigma_1^2(V) + \omega_1[\mu_1(V) - \mu_2(V)]^2\right]}{[\sigma_1^2(V)+\sigma_1(X,V)]^2} \end{aligned} \tag{5}$$

by the definition of γ_1 for external common items.

2 Partial Derivative of $l(x)$ with respect to $\sigma_1^2(X)$

In the linear equating function $l(x)$, $\sigma_s(Y)$ and $\mu_s(Y)$ are not associated with $\sigma_1^2(X)$. Therefore, they can be treated as constants when a partial derivative of $l(x)$ is obtained with respect to $\sigma_1^2(X)$. For the rest of this section, let $A = \sigma_s(Y)$ and $B = \mu_s(Y)$ so that

$$l(x) = A \times [\sigma_s^2(X)]^{-\frac{1}{2}} \times (x - \mu_s(X)) + B. \tag{6}$$

Furthermore, let

$$f = [\sigma_s^2(X)]^{-\frac{1}{2}},$$

and

$$g = (x - \mu_s(X)).$$

Then,

$$l(x) = A \times f \times g + B.$$

Therefore, by the product rule, the partial derivative of $l(x)$ with respect to $\sigma_1^2(X)$ can be obtained as follows:

$$\frac{\partial l(x)}{\partial \sigma_1^2(X)} = A \times \left\{ \frac{\partial f}{\partial \sigma_1^2(X)} \times g + f \times \frac{\partial g}{\partial \sigma_1^2(X)} \right\}, \tag{7}$$

where

$$\begin{aligned} \frac{\partial f}{\partial \sigma_1^2(X)} &= \frac{\partial f}{\partial \sigma_s^2(X)} \times \frac{\partial \sigma_s^2(X)}{\partial \sigma_1^2(X)} \\ &= \left(-\frac{1}{2}\right) \times \left[\sigma_s^2(X)\right]^{-\frac{3}{2}} \times \frac{\partial \sigma_s^2(X)}{\partial \sigma_1^2(X)} \\ &= \frac{-1}{2\sigma_s^3(X)} \frac{\partial \sigma_s^2(X)}{\partial \sigma_1^2(X)}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \frac{\partial g}{\partial \sigma_1^2(X)} &= \frac{\partial g}{\partial \mu_s(X)} \times \frac{\partial \mu_s(X)}{\partial \sigma_1^2(X)} \\ &= (-1) \frac{\partial \mu_s(X)}{\partial \sigma_1^2(X)}. \end{aligned} \tag{9}$$

Consequently, substituting the expressions for partial derivatives from Equations 8 and 9 into Equation 7 gives the partial derivative of $l(x)$ with respect to $\sigma_1^2(X)$ expressed as follows:

$$\frac{\partial l(x)}{\partial \sigma_1^2(X)} = A \times \left\{ \frac{-(x - \mu_s(X))}{2\sigma_s^3(X)} \frac{\partial \sigma_s^2(X)}{\partial \sigma_1^2(X)} - \frac{1}{\sigma_s(X)} \frac{\partial \mu_s(X)}{\partial \sigma_1^2(X)} \right\}. \tag{10}$$

Substituting the expressions for the partial derivatives of $\mu_s(X)$ and $\sigma_s^2(X)$ with respect to $\sigma_1^2(X)$ from Equations 3 and 5 into Equations 10 gives

$$\begin{aligned}
&\frac{\partial l(x)}{\partial \sigma_1^2(X)} \\
&= A \times \left\{ \frac{-Z_x}{2\sigma_s^2(X)} \times \left(1 + \frac{2\omega_2[\sigma_1^2(X) + \sigma_1(X,V)]\left[\sigma_2^2(V) - \sigma_1^2(V) + \omega_1[\mu_1(V) - \mu_2(V)]^2\right]}{[\sigma_1^2(V) + \sigma_1(X,V)]^2}\right) \right. \\
&\qquad \left. - \frac{-\omega_2[\mu_1(V) - \mu_2(V)]}{\sigma_s(X)[\sigma_1^2(V) + \sigma_1(X,V)]} \right\} \\
&= \frac{Z_x A}{\sigma_s^2(X)} \left\{ -\frac{1}{2} - \frac{\omega_2[\sigma_1^2(X) + \sigma_1(X,V)]\left[\sigma_2^2(V) - \sigma_1^2(V) + \omega_1[\mu_1(V) - \mu_2(V)]^2\right]}{[\sigma_1^2(V) + \sigma_1(X,V)]^2} \right\} \\
&\qquad + \frac{\omega_2 A[\mu_1(V) - \mu_2(V)]}{\sigma_s(X)[\sigma_1^2(V) + \sigma_1(X,V)]}
\end{aligned} \tag{11}$$

where $Z_x = (x - \mu_s(X))/\sigma_s(X)$.

Therefore, by substitution $\sigma_s(Y)$ for A in Equation 11, the correct partial derivative of $l(x)$ with respect to $\sigma_1^2(X)$ is

$$\begin{aligned}
&\frac{\partial l(x)}{\partial \sigma_1^2(X)} \\
&= \frac{Z_x \sigma_s(Y)}{\sigma_s^2(X)} \left\{ -\frac{1}{2} - \frac{\omega_2[\sigma_1^2(X) + \sigma_1(X,V)]\left[\sigma_2^2(V) - \sigma_1^2(V) + \omega_1[\mu_1(V) - \mu_2(V)]^2\right]}{[\sigma_1^2(V) + \sigma_1(X,V)]^2} \right\} \\
&\qquad + \frac{\omega_2 \sigma_s(Y)[\mu_1(V) - \mu_2(V)]}{\sigma_s(X)[\sigma_1^2(V) + \sigma_1(X,V)]}.
\end{aligned} \tag{12}$$

Compared to the previous derivative shown in Equation 2, there is no multiplier of -1 in front of the large {} brackets. In other words, compared to the original derivative in Equation 1 by Hanson, Zeng, and Kolen (1993), there are two places where signs are opposite inside the large {} brackets.

3 Example

As an example, one operational test data set was used to compute analytic standard errors using both the original version (Equation 1) (Hanson et al., 1993) and the re-derived version (Equation 12) with and without a normality assumption. The test consists of 36 items with 12 common items as an external anchor. The scores were number-correct scores. In order to demonstrate that the newly derived partial derivative is reasonable, those two different analytic standard errors were compared to standard errors from bootstrapping with 5000 replications. Results are reported in Table 3.1.

As shown in Table 3.1, standard errors using the re-derived version are closer to those for the bootstrap method. For the original version, standard errors are larger than those using the re-derived version for all raw scores. Therefore, the example confirms that the original version of the partial derivative of $l(x)$ with respect to $\sigma_1^2(X)$ should be replaced with the re-derived version in this technical note.

Table 3.1: Standard Errors Using the Original Version (Hanson et al., 1993) and the Re-derived Version with the Normality Assumption (Norm), Without the Normality Assumption (Nonorm), and for the Bootstrap (Boot)

		Norm		Nonorm	
Raw Score	Bootstrap	Original	Re-derived	Original	Re-derived
0	0.31606	0.63822	0.37395	0.60206	0.34209
1	0.30103	0.60231	0.35781	0.56863	0.32839
2	0.28616	0.56663	0.34196	0.53545	0.31499
3	0.27147	0.53126	0.32645	0.50257	0.30192
4	0.25701	0.49624	0.31133	0.47007	0.28923
5	0.24281	0.46167	0.29666	0.43802	0.27696
6	0.22892	0.42764	0.28249	0.40653	0.26519
7	0.21540	0.39430	0.26893	0.37573	0.25397
8	0.20232	0.36184	0.25605	0.34583	0.24339
9	0.18978	0.33052	0.24398	0.31706	0.23353
10	0.17789	0.30069	0.23283	0.28977	0.22448
11	0.16679	0.27286	0.22274	0.26442	0.21635
12	0.15664	0.24767	0.21387	0.24161	0.20925
13	0.14765	0.22604	0.20637	0.22214	0.20328
14	0.14003	0.20905	0.20040	0.20694	0.19854
15	0.13402	0.19792	0.19609	0.19700	0.19513
16	0.12985	0.19364	0.19355	0.19315	0.19311
17	0.12769	0.19668	0.19287	0.19573	0.19253
18	0.12765	0.20671	0.19404	0.20451	0.19340
19	0.12972	0.22279	0.19705	0.21874	0.19570
20	0.13381	0.24371	0.20181	0.23745	0.19939
21	0.13975	0.26836	0.20820	0.25966	0.20438
22	0.14731	0.29580	0.21608	0.28456	0.21059
23	0.15625	0.32533	0.22528	0.31150	0.21791
24	0.16635	0.35642	0.23566	0.34000	0.22623
25	0.17742	0.38871	0.24707	0.36970	0.23544
26	0.18928	0.42191	0.25936	0.40034	0.24546
27	0.20180	0.45584	0.27243	0.43170	0.25617
28	0.21486	0.49033	0.28616	0.46365	0.26751
29	0.22836	0.52528	0.30046	0.49607	0.27939
30	0.24224	0.56060	0.31526	0.52888	0.29174
31	0.25642	0.59622	0.33049	0.56200	0.30451
32	0.27088	0.63210	0.34609	0.59539	0.31765
33	0.28555	0.66819	0.36202	0.62900	0.33111
34	0.30042	0.70446	0.37823	0.66280	0.34486
35	0.31544	0.74089	0.39468	0.69676	0.35887
36	0.33061	0.77744	0.41136	0.73086	0.37310
Overall	0.16921	0.31385	0.23013	0.30142	0.22208

4 References

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