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Chained Linear Equating

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This technical note focuses on chained linear equating for the common-item nonequivalent groups design when scores on Form X are transformed to the scale of Form Y . It is assumed that readers are familiar with the concepts and notation in Kolen and Brennan (2004). It is shown that chained linear observed-score equating can be formulated within the general framework for observed-score equating presented by Kolen and Brennan (2004, p. 122 and elsewhere). In doing so, it is evident that chained linear observed-score equating has certain functional relationships with Tucker and Levine observed-score equating. Also, it is shown that chained linear true-score equating and Levine true-score equating are formally identical.

1 Basic Equations for Chained Linear Equating

The linear equation for equating X to the scale of V in Population 1 (the population that took Form X) is

$$l_{V1}(x) = \left[\mu_1(V) - \frac{\sigma_1(V)}{\sigma_1(X)} \mu_1(X) \right] + \frac{\sigma_1(V)}{\sigma_1(X)}(x) \quad (1)$$

$$= B_{V|x} + A_{V|x}(x), \quad (2)$$

where B is the intercept and A is the slope. The linear equation for equating V to the scale of Y in Population 2 (the population that took Form Y) is

$$l_{Y2}(v) = \left[\mu_2(Y) - \frac{\sigma_2(Y)}{\sigma_2(V)} \mu_2(V) \right] + \frac{\sigma_2(Y)}{\sigma_2(V)}(v) \quad (3)$$

$$= B_{Y|v} + A_{Y|v}(v). \quad (4)$$

The essence of the word ‘‘chained’’ in chained linear equating is the replacement of v in Equation 3 (or 4) with $l_{V1}(x)$ given by Equation 1 (or 2), neglecting the fact that the two equatings are for different populations. That is,

$$\begin{aligned} l_Y(x) &= B_{Y|v} + A_{Y|v}[B_{V|x} + A_{V|x}(x)] \\ &= \left\{ \left[\mu_2(Y) - \frac{\sigma_2(Y)}{\sigma_2(V)} \mu_2(V) \right] + \frac{\sigma_2(Y)}{\sigma_2(V)} \left[\mu_1(V) - \frac{\sigma_1(V)}{\sigma_1(X)} \mu_1(X) \right] \right\} \\ &\quad + \frac{\sigma_2(Y)}{\sigma_2(V)} \frac{\sigma_1(V)}{\sigma_1(X)}(x) \\ &= \left\{ \mu_2(Y) + \frac{\sigma_2(Y)}{\sigma_2(V)} [\mu_1(V) - \mu_2(V)] - \frac{\sigma_2(Y)/\sigma_2(V)}{\sigma_1(X)/\sigma_1(V)} [\mu_1(X)] \right\} \\ &\quad + \frac{\sigma_2(Y)/\sigma_2(V)}{\sigma_1(X)/\sigma_1(V)}(x). \end{aligned} \quad (5)$$

2 Chained Linear Equating in Terms of the Kolen and Brennan (2004) General Observed-Score Equating Equations

In the notation of Kolen and Brennan (2004, p. 122 and elsewhere) the basic equation for observed-score linear equating for the common-item nonequivalent groups design is

$$l_{Y_s}(x) = \frac{\sigma_s(Y)}{\sigma_s(X)}[x - \mu_s(X)] + \mu_s(Y) \quad (6)$$

$$= \left\{ \mu_s(Y) - \frac{\sigma_s(Y)}{\sigma_s(X)}[\mu_s(X)] \right\} + \frac{\sigma_s(Y)}{\sigma_s(X)}(x), \quad (7)$$

where

$$\mu_s(X) = \mu_1(X) - w_2\gamma_1[\mu_1(V) - \mu_2(V)] \quad (8)$$

$$\mu_s(Y) = \mu_2(Y) + w_1\gamma_2[\mu_1(V) - \mu_2(V)] \quad (9)$$

$$\sigma_s^2(X) = \sigma_1^2(X) - w_2\gamma_1^2[\sigma_1^2(V) - \sigma_2^2(V)] + w_1w_2\gamma_1^2[\mu_1(V) - \mu_2(V)]^2 \quad (10)$$

$$\sigma_s^2(Y) = \sigma_2^2(Y) + w_1\gamma_2^2[\sigma_1^2(V) - \sigma_2^2(V)] + w_1w_2\gamma_2^2[\mu_1(V) - \mu_2(V)]^2, \quad (11)$$

with w_1 and w_2 being population weights such that $w_1 + w_2 = 1$. Kolen and Brennan (2004) show that the Tucker and Levine procedures differ only with respect to their γ terms. It is shown below that for chained linear equating, the γ terms are

$$\gamma_1 = \frac{\sigma_1(X)}{\sigma_1(V)}, \quad (12)$$

and

$$\gamma_2 = \frac{\sigma_2(Y)}{\sigma_2(V)}. \quad (13)$$

These results hold for both an internal and an external anchor. We proceed by demonstrating that when Equations 12 and 13 are replaced in Equations 8–11, the slopes and intercepts of Equations 5 and 7 are identical.

2.1 Slope

Replacing Equation 12 in Equation 10 gives

$$\begin{aligned} \sigma_s^2(X) &= \sigma_1^2(X) - w_2 \frac{\sigma_1^2(X)}{\sigma_1^2(V)} [\sigma_1^2(V) - \sigma_2^2(V)] \\ &\quad + w_1w_2 \frac{\sigma_1^2(X)}{\sigma_1^2(V)} [\mu_1(V) - \mu_2(V)]^2 \\ &= \frac{\sigma_1^2(X)}{\sigma_1^2(V)} \{ \sigma_1^2(V) - w_2 [\sigma_1^2(V) - \sigma_2^2(V)] \\ &\quad + w_1w_2 [\mu_1(V) - \mu_2(V)]^2 \}. \end{aligned} \quad (14)$$

Similarly, replacing Equation 13 in Equation 11 gives

$$\begin{aligned}
\sigma_s^2(Y) &= \sigma_2^2(Y) + w_1 \frac{\sigma_2^2(Y)}{\sigma_2^2(V)} [\sigma_1^2(V) - \sigma_2^2(V)] \\
&\quad + w_1 w_2 \frac{\sigma_2^2(Y)}{\sigma_2^2(V)} [\mu_1(V) - \mu_2(V)]^2 \\
&= \frac{\sigma_2^2(Y)}{\sigma_2^2(V)} \{ \sigma_2^2(V) + w_1 [\sigma_1^2(V) - \sigma_2^2(V)] \\
&\quad + w_1 w_2 [\mu_1(V) - \mu_2(V)]^2 \}. \tag{15}
\end{aligned}$$

Note that the first two terms in braces in Equation 15 are

$$\begin{aligned}
&\sigma_2^2(V) + w_1 [\sigma_1^2(V) - \sigma_2^2(V)] \\
&= \sigma_2^2(V) + (1 - w_2) [\sigma_1^2(V) - \sigma_2^2(V)] \\
&= \sigma_2^2(V) + [\sigma_1^2(V) - \sigma_2^2(V)] - w_2 [\sigma_1^2(V) - \sigma_2^2(V)] \\
&= \sigma_1^2(V) - w_2 [\sigma_1^2(V) - \sigma_2^2(V)],
\end{aligned}$$

which is identical to the first two terms in braces in Equation 14. It follows that the slope in Equation 7 is

$$\frac{\sigma_s(Y)}{\sigma_s(X)} = \frac{\sigma_2(Y)/\sigma_2(V)}{\sigma_1(X)/\sigma_1(V)} = \frac{\gamma_2}{\gamma_1}, \tag{16}$$

which is identical to the slope for chained linear equating in Equation 5.

2.2 Intercept

Replacing $\gamma_1 = \sigma_1(X)/\sigma_1(V)$ in Equation 8 gives

$$\mu_s(X) = \mu_1(X) - w_2 \frac{\sigma_1(X)}{\sigma_1(V)} [\mu_1(V) - \mu_2(V)], \tag{17}$$

and replacing $\gamma_2 = \sigma_2(Y)/\sigma_2(V)$ in Equation 9 gives

$$\mu_s(Y) = \mu_2(Y) + w_1 \frac{\sigma_2(Y)}{\sigma_2(V)} [\mu_1(V) - \mu_2(V)]. \tag{18}$$

Using Equations 16–18, it follows that the intercept in Equation 7 is

$$\begin{aligned}
&\mu_s(Y) - \frac{\sigma_s(Y)}{\sigma_s(X)} [\mu_s(X)] \\
&= \left\{ \mu_2(Y) + w_1 \frac{\sigma_2(Y)}{\sigma_2(V)} [\mu_1(V) - \mu_2(V)] \right\} \\
&\quad - \frac{\sigma_2(Y)/\sigma_2(V)}{\sigma_1(X)/\sigma_1(V)} \left\{ \mu_1(X) - w_2 \frac{\sigma_1(X)}{\sigma_1(V)} [\mu_1(V) - \mu_2(V)] \right\} \\
&= \mu_2(Y) + \frac{\sigma_2(Y)}{\sigma_2(V)} [\mu_1(V) - \mu_2(V)] - \frac{\sigma_2(Y)/\sigma_2(V)}{\sigma_1(X)/\sigma_1(V)} [\mu_1(X)], \tag{19}
\end{aligned}$$

which is identical to the intercept for chained linear equating in Equation 5.

2.3 Invariance with Respect to Weights

Note that the results in Equations 16 and 19 for the slope and intercept, respectively, for chained linear equating do not depend on the population weights, w_1 and w_2 . This means that chained linear equating is invariant with respect to population weights, which is *not* the case for Tucker and Levine observed-score equating.

3 Relationships Among Linear Observed-Score Equating Procedures with an Internal Anchor

For an internal anchor, scores on V are included in scores on X and Y . Under these circumstances, there is a relatively simple relationship between the γ terms for the Tucker method, the Levine method under the classical congeneric model, and the chained linear observed-score equating method.

3.1 Form X and Population 1

Consider Form X , which is administered in Population 1. For Tucker equating

$$\gamma_{1T} = \alpha_1(X|V) = \frac{\sigma_1(X, V)}{\sigma_1^2(V)}. \quad (20)$$

For Levine observed-score equating under the classical congeneric model

$$\gamma_{1L} = \frac{1}{\alpha_1(V|X)} = \frac{\sigma_1^2(X)}{\sigma_1(X, V)}. \quad (21)$$

It follows that the γ_1 term for chained linear equating in Equation 12 is

$$\gamma_{1c} = \frac{\sigma_1(X)}{\sigma_1(V)} = \sqrt{\gamma_{1T}\gamma_{1L}}. \quad (22)$$

3.2 Form Y and Population 2

Similarly, for Form Y , which is administered in Population 2,

$$\gamma_{2T} = \alpha_2(Y|V) = \frac{\sigma_2(Y, V)}{\sigma_2^2(V)}, \quad (23)$$

$$\gamma_{2L} = \frac{1}{\alpha_2(V|Y)} = \frac{\sigma_2^2(Y)}{\sigma_2(Y, V)}, \quad (24)$$

and the γ_2 term for chained linear equating in Equation 12 is

$$\gamma_{2c} = \frac{\sigma_2(Y)}{\sigma_2(V)} = \sqrt{\gamma_{2T}\gamma_{2L}}. \quad (25)$$

3.3 Inequalities

As shown by Kolen and Brennan (1987), when $\sigma_1(X, V) > 0$ (as must be the case for equating to be reasonable), $\gamma_{1T} < \gamma_{1L}$. Since the γ terms for chained linear equating are the geometric means of the γ terms for Tucker and Levine observed-score equating, it follows that¹

$$\gamma_{1T} < \gamma_{1c} < \gamma_{1L}. \quad (26)$$

Similarly,²

$$\gamma_{2T} < \gamma_{2c} < \gamma_{2L}. \quad (27)$$

3.4 Interpreting γ Terms

As indicated by Equations 8–11, the γ terms multiply the differences in the first two population moments for scores on the common items. On average, then, larger values for the γ terms cause the associated method to attribute more of the observed raw-score differences in X and Y to population differences and correspondingly less of the observed raw-score differences to form differences. Given the inequalities in Equations 26 and 27, it follows that, on average, the transformation for chained linear equating in Equation 5 should be “between” that for the Tucker and Levine observed-score methods, with the Tucker method attributing more of the observed raw-score differences in X and Y to forms than either of the other two methods. That is, on average, Tucker equivalents should be more extreme than equivalents for chained linear equating, which in turn should be more extreme than equivalents for Levine observed-score equating. Stated more mathematically, on average, $|x - l_Y(x)|$ for Tucker equating should be greater than $|x - l_Y(x)|$ for chained linear equating, which should be greater than $|x - l_Y(x)|$ for Levine observed-score equating.

As a simple example, suppose $w_1 = 1$, which means that the synthetic population is the population that took Form X. Then, from Equations 6 and 9 it is clear that

$$l_Y[\mu_1(X)] = \mu_1(Y) = \mu_2(Y) + \gamma_2[\mu_1(V) - \mu_2(V)].$$

Given the relationship between the γ terms in Equation 27, it follows that

$$l_{YT}[\mu_1(X)] < l_{Yc}[\mu_1(X)] < l_{YL}[\mu_1(X)]$$

when $\mu_1(V) > \mu_2(V)$. Conversely, when $\mu_1(V) < \mu_2(V)$,

$$l_{YT}[\mu_1(X)] > l_{Yc}[\mu_1(X)] > l_{YL}[\mu_1(X)].$$

¹Strictly speaking, if $\rho_1(X, V) = 1$, then all three γ terms are equal.

²Strictly speaking, if $\rho_2(Y, V) = 1$, then all three γ terms are equal.

4 Relationships Among Linear Observed-Score Equating Procedures with an External Anchor

For an external anchor, scores on V are *not* included in scores on X or Y , and expressions for the γ terms for chained linear observed-score equating and Tucker equating are unchanged. That is, for both an internal and an external anchor, Equations 12 and 20 apply for Form X , and Equations 13 and 23 apply for Y . For Levine observed-score equating under the classical-congeneric model, however, the expressions for the γ terms are different for an external anchor. Specifically, for an external anchor

$$\gamma_{1L} = \frac{\sigma_1^2(X) + \sigma_1(X, V)}{\sigma_1^2(V) + \sigma_1(X, V)}, \quad (28)$$

and

$$\gamma_{2L} = \frac{\sigma_2^2(Y) + \sigma_2(Y, V)}{\sigma_2^2(V) + \sigma_2(Y, V)}. \quad (29)$$

4.1 γ_c Terms

For an external anchor,

$$\gamma_{1c} = \sqrt{(\gamma_{1L} - \gamma_{1T}) + \gamma_{1L}\gamma_{1T}}. \quad (30)$$

This can be verified by noting that

$$\gamma_{1c}^2 + \gamma_{1T} = \frac{\sigma_1^2(X) + \sigma_1(X, V)}{\sigma_1^2(V)}$$

and

$$1 + \gamma_{1T} = \frac{\sigma_1^2(V) + \sigma_1(X, V)}{\sigma_1^2(V)},$$

which implies that γ_{1L} in Equation 28 is

$$\gamma_{1L} = \frac{\gamma_{1c}^2 + \gamma_{1T}}{1 + \gamma_{1T}}.$$

Solving for γ_{1c} gives the result in Equation 30. Similarly,

$$\gamma_{2c} = \sqrt{(\gamma_{2L} - \gamma_{2T}) + \gamma_{2L}\gamma_{2T}}. \quad (31)$$

4.2 Inequalities

As shown next, when $\sigma_1(X) > \sigma_1(V)$,

$$\gamma_{1T} < \gamma_{1c} < \gamma_{1L}. \quad (32)$$

Using the transitivity law, this inequality can be proven by demonstrating that $\gamma_{1T} < \gamma_{1c}$ and $\gamma_{1c} < \gamma_{1L}$. First,

$$\gamma_{1T} = \frac{\sigma_1(X, V)}{\sigma_1^2(V)} = \rho_1(X, V) \frac{\sigma_1(X)}{\sigma_1(V)} = \rho_1(X, V) \gamma_{1c} < \gamma_{1c}$$

since $\rho_1(X, V)$ is less than one.³ Second, $\gamma_{1c} < \gamma_{1L}$ means that

$$\frac{\sigma_1(X)}{\sigma_1(V)} < \frac{\sigma_1^2(X) + \sigma_1(X, V)}{\sigma_1^2(V) + \sigma_1(X, V)},$$

which implies that

$$\sigma_1(X) \sigma_1^2(V) + \sigma_1(X) \sigma_1(X, V) < \sigma_1(V) \sigma_1^2(X) + \sigma_1(V) \sigma_1(X, V).$$

Rearranging and combining terms gives

$$\begin{aligned} \sigma_1(X) \sigma_1(X, V) - \sigma_1(V) \sigma_1^2(X) &< \sigma_1(V) \sigma_1(X, V) - \sigma_1(X) \sigma_1^2(V) \\ \sigma_1(X) [\sigma_1(X, V) - \sigma_1(V) \sigma_1(X)] &< \sigma_1(V) [\sigma_1(X, V) - \sigma_1(V) \sigma_1(X)]. \end{aligned}$$

Now, $\rho_1(X, V) < 1$ implies that $\sigma_1(X, V) < \sigma_1(V) \sigma_1(X)$, which in turn means that

$$[\sigma_1(X, V) - \sigma_1(V) \sigma_1(X)] < 0.$$

It follows that

$$\sigma_1(X) > \sigma_1(V).$$

In short, $\gamma_{1c} < \gamma_{1L}$ when $\sigma_1(X) > \sigma_1(V)$. Using a similar proof it can be shown that when $\sigma_2(Y) > \sigma_2(V)$,

$$\gamma_{2T} < \gamma_{2c} < \gamma_{2L}. \quad (33)$$

Using the same type of logic, when $\sigma_1(X) < \sigma_1(V)$, it can be shown that

$$\gamma_{1T} < \gamma_{1L} < \gamma_{1c},$$

and when $\sigma_2(Y) < \sigma_2(V)$

$$\gamma_{2T} < \gamma_{2L} < \gamma_{2c}.$$

These last two inequalities are relatively unrealistic cases, however, because with real data almost always $\sigma_1(X) > \sigma_1(V)$ and $\sigma_2(Y) > \sigma_2(V)$, in which case Equations 32 and 33 apply.

³We disregard the trivial and unrealistic case of $\rho_1(X, V) = 1$

5 Relationship with Levine True-Score Equating

Linear true-score equating under Levine's true-score assumptions when observed-scores are used in place of true scores (see Kolen & Brennan, 2004, Table 4.2) is

$$\begin{aligned}
 l_Y(x) &= \mu_2(Y) + \gamma_2[\mu_1(V) - \mu_2(V)] + (\gamma_2/\gamma_1)[x - \mu_1(X)] \\
 &= \{\mu_2(Y) + \gamma_2[\mu_1(V) - \mu_2(V)] + (\gamma_2/\gamma_1)[\mu_1(X)]\} \\
 &\quad + (\gamma_2/\gamma_1)(x),
 \end{aligned}
 \tag{34}$$

where the two γ terms are ratios of *true-score* standard deviations—namely, $\gamma_1 = \sigma_1(T_X)/\sigma_1(T_V)$ and $\gamma_2 = \sigma_2(T_Y)/\sigma_2(T_V)$. It is evident that Equation 5 for chained linear equating has the same form as Equation 34. The only difference is that for chained linear equating, the γ terms are ratios of *observed-score* standard deviations—namely, $\gamma_1 = \sigma_1(X)/\sigma_1(V)$ and $\gamma_2 = \sigma_2(Y)/\sigma_2(V)$.

It follows that chained linear *true-score* equating is mathematically identical to Levine true-score equating (see Kolen & Brennan, 2004, pp. 115–117, especially Equations 4.38, 4.39, and 4.65). This equivalence extends to Levine true-score equating under the classical congeneric model when the classical congeneric model is applied to both the Levine true-score method and the chained linear true-score method. Consequently, from Hanson's (1991) proof, when chained linear true-score equating is used with observed scores replacing true scores, the resulting equivalents possess the property of first-order equity.

6 References

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