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Unbiased Estimates of Variance Components with Bootstrap Procedures: Detailed Results*

Robert L. Brennan†

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† Robert L. Brennan is E. F. Lindquist Chair in Measurement and Testing and Director, Center for Advanced Studies in Measurement and Assessment (CASMA), 210 Lindquist, College of Education, University of Iowa, Iowa City, IA 52242 (email: robert-brennan@uiowa.edu).

Center for Advanced Studies in
Measurement and Assessment (CASMA)
College of Education
University of Iowa
Iowa City, IA 52242
Tel: 319-335-5439
Web: www.education.uiowa.edu/casma

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Abstract

This paper provides general procedures for obtaining unbiased estimates of variance components for any random-model balanced design under any bootstrap sampling plan, with the focus on designs of the type typically used in generalizability theory. If the only goal is to estimate variance components, there is no advantage to bootstrapping and no need for the procedures discussed in this paper. However, the results reported here are particularly useful when the bootstrap is employed to estimate standard errors of estimated variance components.

For the $p \times i$ design, Wiley (2000) provided formulas for correcting for bias in bootstrap estimates of variance components. In a sense, this paper extends Wiley's results to any design and any bootstrap procedure. There are important differences in approach, however. In particular, in this paper unbiased estimates of variance components are obtained directly for any bootstrap sample through the use of modified expected T-term equations, where T terms are uncorrected sums of squares.

1 Introduction

This paper provides general procedures for obtaining unbiased estimates of variance components for any random-model balanced design under any bootstrap sampling plan, with the focus on designs of the type typically used in generalizability theory. Without any bootstrapping, well-known procedures exist for obtaining unbiased "ANOVA" estimates of variance components for balanced designs (see Searle, 1971, Searle, Casella, & McCulloch, 1992, and Brennan, 2001, among others). However, the procedures discussed here are particularly useful when the bootstrap is employed to estimate standard errors of estimated variance components, as discussed more fully in the companion paper that follows this one.

The most frequently discussed procedure in the literature for estimating standard errors of estimated variance components assumes that score effects are normally distributed. This assumption, however, is often highly suspect in generalizability theory, especially when data are dichotomous. For this reason, nearly 20 years ago Brennan, Harris, and Hanson (1987) studied the possibility of using various bootstrap procedures to estimate standard errors of estimated variance components for the $p \times i$ design with n_p persons and n_i items. (See Brennan, 2001, chap 6, for a summary of Brennan et al., 1987.) Among the procedures they considered were:

- Bootstrap persons (boot–p): draw a random sample of n_p persons with replacement from the n_p persons in the data, keeping the items the same;
- Bootstrap items (boot–i): draw a random sample of n_i items with replacement from the n_i items in the data, keeping the persons the same;
- Bootstrap both persons and items (boot–p, i): draw a random sample of n_p persons with replacement, and a random sample of n_i items with replacement.

For each of these bootstrap procedures, Brennan et al. (1987) estimated the standard error of each of the the variance components estimates as the standard deviation over B replications of the ANOVA-like estimates of the variance component under the particular bootstrap procedure. In a simulation study, they found that the resulting estimated standard errors were often poor, no matter what the nature of the underlying data (normal or dichotomous), and they demonstrated that at least part of the explanation was that the bootstrap ANOVA-like estimates were necessarily biased. Brennan et al. (1987) provided ad hoc correction factors that improved the estimated standard errors, but it was not until thirteen years later that Wiley (2000) provided rigorous derivations of correction factors for the bootstrap estimates of variance components in the $p \times i$ design (see Brennan, 2001, p. 188, for a summary).

Wiley (2000) also provided correction factors for a few bootstrap estimates for the $p \times i \times h$ design, but otherwise the bias in ANOVA-like estimates of variance components under bootstrap procedures has been studied rigorously for the $p \times i$ design, only. In a sense, this paper extends Wiley's (2000) results to any balanced design and any bootstrap procedure. There are important differences in approach, however, as discussed next.

Let $\hat{\sigma}^2(\alpha)$ be the usual ANOVA unbiased estimates of the variance components for the various effects, α. Also, let $\hat{\sigma}^2(\alpha|\lambda)$ be the ANOVA-like estimates that result from a particular replication of boot–λ, where λ is the set of m facets that are bootstrapped; this is what is meant here by the phrase "bootstrap estimates." Wiley (2000), as well as Brennan et al. (1987), viewed the solution to the "bias problem" as one of finding linear functions of bootstrap estimates that gave the $\hat{\sigma}^2(\alpha)$. Also, Wiley (2000) approached the problem by focusing on mean squares to estimate variance components.

By contrast, the approach taken here is to estimate $\sigma^2(\alpha)$ directly from the data for any bootstrap sample using modified versions of the no-bootstrapping expected T-terms equations, $\boldsymbol{E}T(\alpha)$. (T terms are uncorrected sums of squares—see Brennan, 2001, among others). Replacing parameters with estimates, the modified expected T-term equations, $\boldsymbol{E}T(\alpha|\lambda)$, can be solved directly for the $\hat{\sigma}^2(\alpha)$. The principal purpose of this paper is to explain and justify a relatively simple rule for obtaining these modifications.[1] It is also shown that these modified T-term equations can be used to obtain other results such as a general formula for expressing expected mean square equations under boot–λ, as well as equations for $\hat{\sigma}^2(\alpha)$ in terms of bootstrap estimates.

Section 2 provides a detailed review of the $p \times i$ design without bootstrapping, with particular attention given to how T terms can be used to estimate variance components. For the same design, section 3 provides a detailed discussion of derivations that give expected T-term equations for boot–p, boot–i, and boot–p, i. Section 4 extends these results to the $p \times (i\!:\!h)$ design and each of the seven possible bootstrap procedures. Finally, section 5 provides general procedures and formulas for any balanced design and any bootstrap procedure.

Throughout this paper no distinction is made between measurement facets and the "facet" that represents the objects of measurement. That is, here there are as many facets as there are indexes used to represent the design. Also, the model under consideration is always assumed to be random. To assist the reader, there is some purposeful repetition of results across sections.

2 The $p \times i$ Design Without Bootstrapping

The principal purpose of this section is to describe how T terms can be used to estimate variance components for the $p \times i$ design. Subsequently, it is shown how mean squares are related to T terms and, hence, how mean squares can be used to estimate variance components. Whether one uses T terms or mean squares, the resulting estimates are equal since they are based on the same quadratic forms, and these estimates are BQUE (best quadratic unbiased estimates). There is nothing novel about this section relative to already available

[1] Section 5.3 provides a succinct statement of the rule, and Appendices B and C illustrate it.

literature such as Searle (1971), Searle et al. (1992), and Brennan (2001). This section is included only to establish the notational conventions and groundwork for using T terms to estimate variance components for the $p \times i$ design. The results presented here are extended to bootstrap sampling procedures in section 3.

2.1 T term for p

The T term for persons is

$$\begin{aligned}
T(p) &= n_i \sum_{p=1}^{n_p} \overline{X}_p^2 \\
&= n_i \sum_{p=1}^{n_p} \left[\frac{1}{n_i} \sum_{i=1}^{n_i} (\mu + \nu_p + \nu_i + \nu_{pi}) \right]^2 \\
&= n_i \sum_{p=1}^{n_p} (\mu + \nu_p + \nu_I + \nu_{pI})^2 \\
&= n_i \sum_{p=1}^{n_p} (\mu^2 + \nu_p^2 + \nu_I^2 + \nu_{pI}^2 \\
&\qquad + 2\mu\, \nu_p + 2\mu\, \nu_I + 2\mu\, \nu_{pI} + 2\nu_p\, \nu_I + 2\nu_p\, \nu_{pI} + 2\nu_I \nu_{pI}), \quad (1)
\end{aligned}$$

where uppercase I designates the mean over n_i items. Letting $\boldsymbol{E}$ stand for expected value, the manner in which score effects are defined in generalizability theory *guarantees* that $\boldsymbol{E}\nu_\alpha = 0$ for any of the indices in α, and $\boldsymbol{E}\nu_\alpha\nu_\beta = 0$ when α and β do not share any indexes in common. In addition, when there is overlap in the α and β indexes, it is usually assumed that $\boldsymbol{E}\nu_\alpha\nu_\beta = 0$. Under these conditions, it is obvious that the expected value of $T(p)$ in Equation 1 is

$$\begin{aligned}
\boldsymbol{E}T(p) &= n_i \boldsymbol{E} \sum_{p=1}^{n_p} (\mu^2 + \nu_p^2 + \nu_I^2 + \nu_{pI}^2) \\
&= n_p n_i\, \mu^2 + n_p n_i\, \sigma^2(p) + n_p n_i \boldsymbol{E} \left(\frac{1}{n_i} \sum_{i=1}^{n_i} \nu_i \right)^2 + n_p n_i \boldsymbol{E} \left(\frac{1}{n_i} \sum_{i=1}^{n_i} \nu_{pi} \right)^2 \\
&= n_p n_i\, \mu^2 + n_p n_i\, \sigma^2(p) + \frac{n_p}{n_i} \boldsymbol{E} \left(\sum_{i=1}^{n_i} \nu_i \right)^2 + \frac{n_p}{n_i} \boldsymbol{E} \left(\sum_{i=1}^{n_i} \nu_{pi} \right)^2. \quad (2)
\end{aligned}$$

The kernel of the third term of Equation 2 is

$$\begin{aligned}
\boldsymbol{E} \left(\sum_{i=1}^{n_i} \nu_i \right)^2 &= \boldsymbol{E} \left(\sum_i \nu_i^2 + \sum_{i \neq i'}\sum \nu_i \nu_{i'} \right) \\
&= \sum_i \boldsymbol{E}\nu_i^2 + \sum_{i \neq i'}\sum \boldsymbol{E}(\nu_i \nu_{i'})
\end{aligned}$$

$$= \quad n_i\,\sigma^2(i) + n_i(n_i-1)\boldsymbol{E}(\nu_i\nu_{i'}). \tag{3}$$

Similarly, the kernel of the fourth term of Equation 2 is

$$\boldsymbol{E}\left(\sum_{i=1}^{n_i}\nu_{pi}\right)^2 \quad = \quad n_i\,\sigma^2(pi) + n_i(n_i-1)\boldsymbol{E}(\nu_{pi}\nu_{pi'}). \tag{4}$$

For the random model with no bootstrapping and $i \neq i'$, $\boldsymbol{E}(\nu_i\nu_{i'}) = \boldsymbol{E}(\nu_{pi}\nu_{pi'}) = 0$, which means that the third term of Equation 2 is $n_p\,\sigma^2(i)$, and the fourth term is $n_p\,\sigma^2(pi)$. It follows that

$$\boldsymbol{E}T(p) = n_p n_i\,\mu^2 + n_p n_i\,\sigma^2(p) + n_p\,\sigma^2(i) + n_p\,\sigma^2(pi). \tag{5}$$

2.2 $\boldsymbol{T}$ term for $\boldsymbol{i}$

The T term for items is

$$T(i) = n_p \sum_{i=1}^{n_i} \overline{X}_i^2 = n_p \sum_{i=1}^{n_i} (\mu + \nu_P + \nu_i + \nu_{Pi})^2\,, \tag{6}$$

where uppercase P designates the mean over n_p persons. Since the $p \times i$ design involves a clear symmetry between p and i, a derivation similar to that for $\boldsymbol{E}T(p)$ gives

$$\boldsymbol{E}T(i) \quad = \quad n_p n_i\,\mu^2 + \frac{n_i}{n_p}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\nu_p\right)^2 + n_p n_i \sigma^2(i) + \frac{n_i}{n_p}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\nu_{pi}\right)^2. \tag{7}$$

The kernel of the second term of Equation 7 is

$$\boldsymbol{E}\left(\sum_{p=1}^{n_p}\nu_p\right)^2 \quad = \quad n_p\,\sigma^2(p) + n_p(n_p-1)\boldsymbol{E}(\nu_p\nu_{p'}). \tag{8}$$

Similarly, the kernel of the fourth term of Equation 7 is

$$\boldsymbol{E}\left(\sum_{p=1}^{n_p}\nu_{pi}\right)^2 \quad = \quad n_p\,\sigma^2(pi) + n_p(n_p-1)\boldsymbol{E}(\nu_{pi}\nu_{p'i}). \tag{9}$$

For the random model (without bootstrapping), $\boldsymbol{E}(\nu_p\nu_{p'}) = \boldsymbol{E}(\nu_{pi}\nu_{p'i}) = 0$, which leads to

$$\boldsymbol{E}T(i) = n_p n_i\,\mu^2 + n_i\,\sigma^2(p) + n_p n_i\,\sigma^2(i) + n_i\,\sigma^2(pi). \tag{10}$$

2.3 T term for pi

The T term for pi is

$$T(pi) = \sum_{p=1}^{n_p}\sum_{i=1}^{n_i}\overline{X}_{pi}^2 = \sum_{p=1}^{n_p}\sum_{i=1}^{n_i}(\mu+\nu_p+\nu_i+\nu_{pi})^2 \tag{11}$$

with the expected value being

$$\boldsymbol{E}T(pi) = n_p n_i\,\mu^2 + n_p n_i\,\sigma^2(p) + n_p n_i\,\sigma^2(i) + n_p n_i\,\sigma^2(pi). \tag{12}$$

2.4 T term for μ

The T term for μ is

$$T(\mu) = n_p n_i \overline{X}^2 = n_p n_i(\mu+\nu_P+\nu_I+\nu_{PI})^2, \tag{13}$$

with the expected value being

$$\begin{aligned}\boldsymbol{E}T(\mu) &= n_p n_i\,\mu^2 + n_p n_i \boldsymbol{E}\nu_P^2 + n_p n_i \boldsymbol{E}\nu_I^2 + n_p n_i \boldsymbol{E}\nu_{PI}^2 \\ &= n_p n_i\,\mu^2 + \frac{n_i}{n_p}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\nu_p\right)^2 + \frac{n_p}{n_i}\boldsymbol{E}\left(\sum_{i=1}^{n_i}\nu_i\right)^2 \\ &\quad + \frac{1}{n_p n_i}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\sum_{i=1}^{n_i}\nu_{pi}\right)^2. \end{aligned} \tag{14}$$

The kernel in the second term is given by Equation 8, the kernel of the third term is given by Equation 3, and the kernel in the fourth term is

$$\begin{aligned}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\sum_{i=1}^{n_i}\nu_{pi}\right)^2 &= \boldsymbol{E}\left(\sum_p\sum_i \nu_{pi}^2 + \sum\sum\sum\sum_{(p\neq p')\cup(i\neq i')}\nu_{pi}\nu_{p'i'}\right) \\ &= \sum_p\sum_i \boldsymbol{E}\nu_{pi}^2 + \sum\sum\sum\sum_{(p\neq p')\cup(i\neq i')}\boldsymbol{E}(\nu_{pi}\nu_{p'i'}) \\ &= n_p n_i\,\sigma^2(pi) + \sum\sum\sum\sum_{(p\neq p')\cup(i\neq i')}\boldsymbol{E}(\nu_{pi}\nu_{p'i'}), \end{aligned} \tag{15}$$

where the condition under the quadruple summation means that it is *not* true that both $p = p'$ and $i = i'$. For the random model with no bootstrapping, when $p \neq p'$ and $i \neq i'$,

$$\boldsymbol{E}(\nu_p\nu_{p'}) = \boldsymbol{E}(\nu_i\nu_{i'}) = \boldsymbol{E}(\nu_{pi}\nu_{pi'}) = \boldsymbol{E}(\nu_{pi}\nu_{p'i}) = \boldsymbol{E}(\nu_{pi}\nu_{p'i'}) = 0.$$

It follows that

$$\boldsymbol{E}T(\mu) = n_p n_i\,\mu^2 + n_i\,\sigma^2(p) + n_p\,\sigma^2(i) + \sigma^2(pi). \tag{16}$$

2.5 Using T terms to Estimate $\sigma^2(\alpha)$

Equations 5, 10, 12, and 16 constitute the set of random model (no bootstrapping) expected T terms:

$$\begin{aligned} \boldsymbol{E}T(p) &= n_p n_i\,\mu^2 + n_p\sigma^2(pi) + n_p\sigma^2(i) + n_p n_i\sigma^2(p) \\ \boldsymbol{E}T(i) &= n_p n_i\,\mu^2 + n_i\sigma^2(pi) + n_p n_i\sigma^2(i) + n_i\sigma^2(p) \\ \boldsymbol{E}T(pi) &= n_p n_i\,\mu^2 + n_p n_i\sigma^2(pi) + n_p n_i\sigma^2(i) + n_p n_i\sigma^2(p) \\ \boldsymbol{E}T(\mu) &= n_p n_i\,\mu^2 + \sigma^2(pi) + n_p\sigma^2(i) + n_i\sigma^2(p). \end{aligned} \tag{17}$$

If we replace the $\sigma^2(\alpha)$ with $\hat{\sigma}^2(\alpha)$, and we replace the $\boldsymbol{E}T(\alpha)$ with actual T terms, then we have four equations in four unknowns. These equations can be solved for unbiased estimates of the $\sigma^2(\alpha)$. A bit of algebra gives

$$\begin{aligned} \hat{\sigma}^2(p) &= \frac{1}{n_i}\left[\frac{n_iT(p) - n_iT(\mu) - T(pi) + T(i)}{(n_p-1)(n_i-1)}\right] \\ \hat{\sigma}^2(i) &= \frac{1}{n_p}\left[\frac{n_pT(i) - n_pT(\mu) - T(pi) + T(p)}{(n_p-1)(n_i-1)}\right] \\ \hat{\sigma}^2(pi) &= \frac{T(pi) - T(p) - T(i) + T(\mu)}{(n_p-1)(n_i-1)}. \end{aligned} \tag{18}$$

2.6 Using Mean Squares to Estimate $\sigma^2(\alpha)$

The mean squares are:

$$\begin{aligned} MS(p) &= \frac{T(p) - T(\mu)}{n_p - 1} \\ MS(i) &= \frac{T(i) - T(\mu)}{n_i - 1} \\ MS(pi) &= \frac{T(pi) - T(p) - T(i) + T(\mu)}{(n_p-1)(n_i-1)}, \end{aligned} \tag{19}$$

and it is well known that the expected mean square equations are:

$$\begin{aligned} \boldsymbol{E}MS(p) &= \sigma^2(pi) + n_i\,\sigma^2(p) \\ \boldsymbol{E}MS(i) &= \sigma^2(pi) + n_p\,\sigma^2(i) \\ \boldsymbol{E}MS(pi) &= \sigma^2(pi). \end{aligned} \tag{20}$$

If we replace the $\sigma^2(\alpha)$ with $\hat{\sigma}^2(\alpha)$ and we replace the $\boldsymbol{E}MS(\alpha)$ with actual mean squares, then it is easy to show that

$$\begin{aligned} \hat{\sigma}^2(p) &= \frac{MS(p) - MS(pi)}{n_i} \\ \hat{\sigma}^2(i) &= \frac{MS(i) - MS(pi)}{n_p} \\ \hat{\sigma}^2(pi) &= MS(pi). \end{aligned} \tag{21}$$

In Equation Sets 18 and 21, the estimators of variance components are the same, as they must be since they are based on the same quadratic forms.

3 The $p \times i$ Design With Bootstrapping

In this section, we extend the basic notions discussed in section 2 to the bootstrap sampling procedures boot–p (bootstrapping persons only), boot–i (bootstrapping items only), and boot–p, i (independently bootstrapping both persons and items).[2] Bootstrapping persons means that n_p persons are randomly sampled with replacement from the n_p persons in the data set. So, the same person could appear more than once in a particular bootstrap replication, but the items would be the same. Bootstrapping items means that n_i items are randomly sampled with replacement from the n_i items in the data set. Bootstrapping both persons and items means that n_p persons are randomly sampled with replacement from the n_p persons in the data set and, independently, n_i items are randomly sampled with replacement from the n_i items in the data set.

For these bootstrap procedures, the results in Appendix A were originally reported by Wiley (2000). These results are expressions for unbiased estimates of variance components in terms of the biased estimates that result from bootstrapping. By contrast, the focus here is on deriving unbiased estimates of variance components using T terms from a bootstrap replication. There is no particular advantage to doing so for the $p \times i$ design; the advantage is that the process to be discussed extends quite easily to any bootstrap procedure and any balanced design of the type typically encountered in generalizability theory. That is the focus of sections 4 and 5.

Equations 2, 7, 12, and 14, are general forms of the expected T terms for the $p \times i$ design. They are repeated next for ease of reference and to highlight their general form.

$$\begin{aligned}
\boldsymbol{E}T(p) &= n_p n_i\,\mu^2 + n_p n_i\,\sigma^2(p) + \frac{n_p}{n_i}\boldsymbol{E}\left(\sum_{i=1}^{n_i}\nu_i\right)^2 + \frac{n_p}{n_i}\boldsymbol{E}\left(\sum_{i=1}^{n_i}\nu_{pi}\right)^2\\
\boldsymbol{E}T(i) &= n_p n_i\,\mu^2 + \frac{n_i}{n_p}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\nu_p\right)^2 + n_p n_i \sigma^2(i) + \frac{n_i}{n_p}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\nu_{pi}\right)^2\\
\boldsymbol{E}T(pi) &= n_p n_i\,\mu^2 + n_p n_i\,\sigma^2(p) + n_p n_i\,\sigma^2(i) + n_p n_i\,\sigma^2(pi)\\
\boldsymbol{E}T(\mu) &= n_p n_i\,\mu^2 + \frac{n_i}{n_p}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\nu_p\right)^2 + \frac{n_p}{n_i}\boldsymbol{E}\left(\sum_{i=1}^{n_i}\nu_i\right)^2\\
&\quad + \frac{1}{n_p n_i}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\sum_{i=1}^{n_i}\nu_{pi}\right)^2.
\end{aligned}$$

As shown in section 2, under the random model with no bootstrapping,

$$\frac{n_p}{n_i}\boldsymbol{E}\left(\sum_{i=1}^{n_i}\nu_i\right)^2 = n_p\,\sigma^2(i), \tag{22}$$

[2]Brennan et al. (1987) and Wiley (2000) also discuss bootstrapping persons, items, and residuals (boot–p, i, r). This possibility is not considered here, although it certainly could be.

$$\frac{n_p}{n_i}\boldsymbol{E}\left(\sum_{i=1}^{n_i}\nu_{pi}\right)^2 = n_p\,\sigma^2(pi), \tag{23}$$

$$\frac{n_i}{n_p}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\nu_{p}\right)^2 = n_i\sigma^2(p), \tag{24}$$

$$\frac{n_i}{n_p}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\nu_{pi}\right)^2 = n_i\sigma^2(pi), \tag{25}$$

$$\frac{1}{n_p n_i}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\sum_{i=1}^{n_i}\nu_{pi}\right)^2 = \sigma^2(pi), \tag{26}$$

and the resulting expected T-term equations are given by Equation Set 17.

Under each of the possible bootstrapping procedures, however, not all of Equations 22–26 hold. We begin with boot–i.

3.1 Boot–i

Under boot–i, Equations 22, 23, and 26 do not hold. The corresponding equations under boot–i are derived next.

3.1.1 Results for $(\sum_i \nu_i)^2$

In the general expression for $\boldsymbol{E}T(p)$ on page 7, the third term is $(n_p/n_i)\boldsymbol{E}(\sum_i \nu_i)^2$ where, as noted previously (see Equation 3),

$$\boldsymbol{E}\left(\sum_{i=1}^{n_i}\nu_{i}\right)^2 = n_i\,\sigma^2(i) + n_i(n_i-1)\boldsymbol{E}(\nu_i\nu_{i'}).$$

When items are sampled independently from an infinite universe $\boldsymbol{E}(\nu_i\nu_{i'}) = 0$. Under boot–$i$, however, $\boldsymbol{E}(\nu_i\nu_{i'}) = \sigma^2(i)$ whenever i and i' are the same items, which can certainly happen since items are sampled with replacement. Consider Table 1 which provides three of the possible 256 replications for boot–i when $n_i = 4$. The diagonal cells (designated with the symbol $*$) necessarily contain $\sigma^2(i)$ with or without bootstrapping. They contribute to the first term in the above equation. Clearly, over all replications, the expected number of times the diagonal cells contain $\sigma^2(i)$ is precisely n_i; i.e.,

$$\boldsymbol{E}(\text{number of diagonal matches}) = n_i. \tag{27}$$

For each of the three replications, the off-diagonal cells in Table 1 for which $\boldsymbol{E}(\nu_i\nu_{i'}) = \sigma^2(i)$ are designated with the symbol i, reflecting the fact that they are non-zero solely as a result of bootstrapping items. The question is, “How frequently is it expected that this will happen?” Under sampling with replacement from a finite pool of n_i items, the probability that a specific item is sampled twice is $1/n_i^2$. Since there are n_i items, the probability of a match in

Table 1: Illustration of when $\boldsymbol{E}\nu_i\nu_{i'} = \sigma^2(i)$ for Three Possible Replications of Boot–i with $n_i = 4$

	i_1	i_3	i_3	i_4
i_1	*			
i_3		*	i	
i_3		i	*	
i_4				*

	i_1	i_1	i_3	i_3
i_1	*	i		
i_1	i	*		
i_3			*	i
i_3			i	*

	i_1	i_1	i_1	i_3
i_1	*	i	i	
i_1	i	*	i	
i_1	i	i	*	
i_3				*

two sampled items is $n_i(1/n_i^2) = 1/n_i$. Now, given n_i items, there are $n_i(n_i - 1)$ opportunities for a match, because there are $n_i(n_i - 1)$ off-diagonal elements in any replication. It follows that

$$\boldsymbol{E}(\text{number of off-diagonal matches}) = \frac{1}{n_i}\, n_i(n_i - 1) = n_i - 1. \tag{28}$$

Adding Equations 27 and 28 gives

$$\boldsymbol{E}(\text{number of matches}) = 2n_i - 1, \tag{29}$$

which means that

$$\boldsymbol{E}\left(\sum_{i=1}^{n_i} \nu_i\right)^2 = (2n_i - 1)\,\sigma^2(i), \tag{30}$$

and

$$\frac{n_p}{n_i}\,\boldsymbol{E}\left(\sum_{i=1}^{n_i} \nu_i\right)^2 = n_p s_i\,\sigma^2(i), \tag{31}$$

where

$$s_i = \frac{2n_i - 1}{n_i}. \tag{32}$$

In other words, under boot–i, the no-bootstrapping result in Equation 22, $n_p\,\sigma^2(i)$, is multiplied by s_i. Equation 31 also applies to the ν_i term in $\boldsymbol{E}T(\mu)$.

3.1.2 Results for $(\sum_i \nu_{pi})^2$

The same basic logic applies to the ν_{pi} term in the general expression for $\boldsymbol{E}T(p)$ on page 7; i.e.,

$$\frac{n_p}{n_i}\,\boldsymbol{E}\left(\sum_{i=1}^{n_i} \nu_{pi}\right)^2 = n_p s_i\,\sigma^2(pi). \tag{33}$$

Table 2: Illustration of when $\boldsymbol{E}\nu_{pi}\nu_{p'i'} = \sigma^2(pi)$ for one Possible Replication for boot–i with $n_p = 3$ and $n_i = 4$

		p_1				p_2				p_3			
		i_1	i_3	i_3	i_4	i_1	i_3	i_3	i_4	i_1	i_3	i_3	i_4
p_1	i_1	*											
	i_3		*	i									
	i_3		i	*									
	i_4				*								
p_2	i_1					*							
	i_3						*	i					
	i_3						i	*					
	i_4								*				
p_3	i_1									*			
	i_3										*	i	
	i_3										i	*	
	i_4												*

3.1.3 Results for $(\sum_p \sum_i \nu_{pi})^2$

In addition, as indicated next, when only items are bootstrapped, the ν_{pi} term in $\boldsymbol{E}T(\mu)$ on page 7 is

$$\frac{1}{n_p n_i}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\sum_{i=1}^{n_i}\nu_{pi}\right)^2 \quad = \quad s_i\,\sigma^2(pi). \tag{34}$$

This is illustrated in Table 2 for a single replication of boot–i with $n_p = 3$ and $n_i = 4$. There are two crucial points to note:

- since only items are bootstrapped, each large off-diagonal cell involves a different pair of persons, and in such cases $\boldsymbol{E}\nu_{pi}\nu_{p'i} = \boldsymbol{E}\nu_{pi}\nu_{p'i'} = 0$; and
- each of the n_p diagonal cells has the same form as that in the derivation of $\boldsymbol{E}(\sum_i \nu_i)^2$ in Equation 30.

It follows that

$$\frac{1}{n_p n_i}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\sum_{i=1}^{n_i}\nu_{pi}\right)^2 = \frac{1}{n_i}\boldsymbol{E}\left(\sum_{i=1}^{n_i}\nu_{pi}\right)^2 = s_i\,\sigma^2(pi). \tag{35}$$

For a more mathematically rigorous proof, consider the following expansion of the kernel of the left side of Equation 34:

$$\boldsymbol{E}\left(\sum_{p=1}^{n_p}\sum_{i=1}^{n_i}\nu_{pi}\right)^2 \quad = \quad \sum_{p\,\cap\,i}\sum \boldsymbol{E}\nu_{pi}^2$$

$$
\begin{aligned}
&+\sum\sum_{p\,\cap(i\neq i')}\sum \boldsymbol{E}(\nu_{pi}\nu_{pi'}) \\
&+\sum\sum_{(p\neq p')\cap\, i}\sum \boldsymbol{E}(\nu_{pi}\nu_{p'i}) \\
&+\sum\sum_{(p\neq p')\cap(i\neq i')}\sum\sum \boldsymbol{E}(\nu_{pi}\nu_{p'i'}), \qquad (36)
\end{aligned}
$$

where $\cap$ means "and." The last two terms are zero because under boot–i, $\boldsymbol{E}\nu_{pi}\nu_{p'i} = \boldsymbol{E}\nu_{pi}\nu_{p'i'} = 0$. The first two terms combined are $n_p\boldsymbol{E}(\sum_i \nu_{pi})^2$, which means that Equation 35 holds.

3.1.4 Using T Terms to Estimate $\sigma^2(\alpha)$

In short, the set of expected T terms under boot–i is:

$$
\begin{aligned}
\boldsymbol{E}T(p|i) &= n_p n_i\,\mu^2 + n_p s_i\,\sigma^2(pi) + n_p s_i\,\sigma^2(i) + n_p n_i\,\sigma^2(p) \\
\boldsymbol{E}T(i|i) &= n_p n_i\,\mu^2 + n_i\sigma^2(pi) + n_p n_i\,\sigma^2(i) + n_i\,\sigma^2(p) \\
\boldsymbol{E}T(pi|i) &= n_p n_i\,\mu^2 + n_p n_i\,\sigma^2(pi) + n_p n_i\,\sigma^2(i) + n_p n_i\,\sigma^2(p) \\
\boldsymbol{E}T(\mu|i) &= n_p n_i\,\mu^2 + s_i\,\sigma^2(pi) + n_p s_i\,\sigma^2(i) + n_i\,\sigma^2(p).
\end{aligned} \qquad (37)
$$

Note that the expected T terms on the left are conditional on boot–i, whereas the variance components on the right have nothing to do with bootstrapping. Clearly, these four equations can be used to estimate the variance components with respect to the $T(\alpha|i)$. Doing so gives

$$
\begin{aligned}
\hat{\sigma}^2(p) &= \frac{n_i T(p\,|i) - n_i T(\mu|i) - s_i T(pi|i) + s_i T(i|i)}{(n_p-1)(n_i-1)^2} \\
\hat{\sigma}^2(i) &= \frac{n_p T(i|i) - n_p T(\mu|i) - T(pi|i) + T(p\,|i)}{t_i n_p (n_p-1)(n_i-1)} \\
\hat{\sigma}^2(pi) &= \frac{T(pi|i) - T(p\,|i) - T(i|i) + T(\mu|i)}{t_i(n_p-1)(n_i-1)}.
\end{aligned} \qquad (38)
$$

3.1.5 Using Mean Squares to Estimate $\sigma^2(\alpha)$

The mean squares for boot–i are

$$
\begin{aligned}
MS(p\,|i) &= \frac{T(p|i) - T(\mu|i)}{n_p - 1} \\
MS(i|i) &= \frac{T(i|i) - T(\mu|i)}{n_i - 1} \\
MS(pi|i) &= \frac{T(pi|i) - T(p|i) - T(i|i) + T(\mu|i)}{(n_p-1)(n_i-1)}.
\end{aligned} \qquad (39)
$$

Given Equations Set 37, the expected mean squares are

$$
\boldsymbol{E}MS(p\,|i) = s_i\,\sigma^2(pi) + n_i\,\sigma^2(p)
$$

$$\begin{aligned} \boldsymbol{E}MS(i|i) &= t_i\,\sigma^2(pi) + t_i n_p\,\sigma^2(i) \\ \boldsymbol{E}MS(pi|i) &= t_i\,\sigma^2(pi), \end{aligned} \tag{40}$$

where

$$t_i = \frac{n_i - s_i}{n_i - 1} = \frac{n_i - 1}{n_i}. \tag{41}$$

Note for later reference that

$$s_i - t_i = 1,$$

and

$$t_i n_i = n_i - 1.$$

It is easy to use Equation Set 40 to obtain expressions for $\hat{\sigma}^2(\alpha)$ in terms of the mean squares that result from a bootstrap replication, $MS(\alpha|i)$:

$$\begin{aligned} \hat{\sigma}^2(p) &= \frac{t_i MS(p|i) - s_i MS(pi|i)}{t_i n_i} \\ \hat{\sigma}^2(i) &= \frac{MS(i|i) - MS(pi|i)}{t_i n_p} \\ \hat{\sigma}^2(pi) &= \frac{MS(pi|i)}{t_i}. \end{aligned} \tag{42}$$

These equations give identical results to those in Equation Set 38, as they must since both sets of equations are based on the same fundamental set of quadratic forms.

3.1.6 Expressions for $\hat{\sigma}^2(\alpha)$ in terms of $\hat{\sigma}^2(\alpha|i)$

Sometimes it is useful to have expressions for $\hat{\sigma}^2(\alpha)$ in terms of the boot–i estimators, $\hat{\sigma}^2(\alpha|i)$. To obtain such expressions, we use Equation Set 40 in conjunction with the boot-i version of Equation Set 21, namely,

$$\begin{aligned} \hat{\sigma}^2(p|i) &= \frac{MS(p|i) - MS(pi|i)}{n_i} \\ \hat{\sigma}^2(i|i) &= \frac{MS(i|i) - MS(pi|i)}{n_p} \\ \hat{\sigma}^2(pi|i) &= MS(pi|i). \end{aligned} \tag{43}$$

Note the difference between Equation Sets 42 and 43. The former are expressions for the unbiased estimators of variance components in terms of mean squares that result from boot–i. The latter are the *biased* estimators of variance components that result from applying the usual mean-square formulas when the sampling plan is boot–i.

Starting at the bottom of Equation Set 43, and recalling the results in Equation Set 40,

$$\hat{\sigma}^2(pi|i) = MS(pi|i) = t_i\hat{\sigma}^2(pi),$$

which means that

$$\hat{\sigma}^2(pi) = \frac{\hat{\sigma}^2(pi|i)}{t_i}.$$

Then,

$$\hat{\sigma}^2(i|i) = \frac{MS(i|i) - MS(pi|i)}{n_p} = \frac{[t_i\,\hat{\sigma}^2(pi) + t_i n_p\,\hat{\sigma}^2(i)] - t_i\,\sigma^2(pi)}{n_p} = t_i\,\hat{\sigma}^2(i),$$

which means that

$$\hat{\sigma}^2(i) = \frac{\hat{\sigma}^2(i|i)}{t_i}.$$

Finally,

$$\begin{aligned}
\hat{\sigma}^2(p|i) &= \frac{MS(p|i) - MS(pi|i)}{n_i} \\
&= \frac{[s_i\,\hat{\sigma}^2(pi) + n_i\,\hat{\sigma}^2(p)] - t_i\,\hat{\sigma}^2(pi)}{n_i} \\
&= \hat{\sigma}^2(p) + \frac{\hat{\sigma}^2(pi)}{n_i} \\
&= \hat{\sigma}^2(p) + \frac{\hat{\sigma}^2(pi|i)}{t_i n_i} \\
&= \hat{\sigma}^2(p) + \frac{\hat{\sigma}^2(pi|i)}{n_i - 1},
\end{aligned}$$

which means that

$$\hat{\sigma}^2(p) = \hat{\sigma}^2(p|i) - \frac{\hat{\sigma}^2(pi|i)}{n_i - 1}.$$

In short,

$$\begin{aligned}
\hat{\sigma}^2(p) &= \hat{\sigma}^2(p|i) - \frac{\hat{\sigma}^2(pi|i)}{n_i - 1} \\
\hat{\sigma}^2(i) &= \frac{\hat{\sigma}^2(i|i)}{t_i} \\
\hat{\sigma}^2(pi) &= \frac{\hat{\sigma}^2(pi|i)}{t_i}.
\end{aligned} \tag{44}$$

3.1.7 Summary

Clearly, deriving all the results presented here involves a formidable amount of algebra. Still, there is a relatively simple solution to the problem of obtaining unbiased estimates of the variance components, $\hat{\sigma}^2(\alpha)$, under boot–i, as noted below.

- Write out the usual (i.e., no bootstrapping) expected T-term equations given by Equation Set 17.

- For those T terms that do *not* contain i, locate the variance components (in the expected T-term equations) that *do* contain i, and multiple their coefficients by s_i in Equation 32.
- Use the resulting set of boot–i T-term equations to obtain the $\hat{\sigma}^2(\alpha)$. This can be a bit challenging to do by hand, but it is easily accomplished with computerized matrix algebra routines.

The remainder of this paper will extend and exploit these basic ideas to any bootstrap procedure and any balanced design.

For boot–i we have also derived:

- expected boot–i mean squares in terms of variance components (Equation Set 40),
- estimated variance components in terms of boot–i mean squares (Equation Set 42), and
- estimated variance components in terms of estimated variance components that result directly from boot–i (Equation Set 44).

These additional sets of equations can be useful, but they are not necessary to derive estimated variance components. The expected T-term equations are sufficient for that purpose.

3.2 Boot–p

The symmetry between p and i in the $p \times i$ design can be exploited to obtain results for boot–p. Doing so gives

$$\boldsymbol{E}\left(\sum_{p=1}^{n_p} \nu_p\right)^2 = (2n_p - 1)\,\sigma^2(p), \tag{45}$$

which means that

$$\frac{n_i}{n_p}\boldsymbol{E}\left(\sum_{p=1}^{n_p} \nu_p\right)^2 = n_i s_p\,\sigma^2(p). \tag{46}$$

Also,

$$\frac{n_i}{n_p}\boldsymbol{E}\left(\sum_{p=1}^{n_p} \nu_{pi}\right)^2 = n_i s_p\,\sigma^2(pi), \tag{47}$$

and

$$\frac{1}{n_p n_i}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\sum_{i=1}^{n_i} \nu_{pi}\right)^2 = s_p\,\sigma^2(pi). \tag{48}$$

It follows that

$$\begin{aligned}
\boldsymbol{E}T(p|p) &= n_p n_i\,\mu^2 + \quad n_p\sigma^2(pi) + \quad n_p\,\sigma^2(i) + n_p n_i\,\sigma^2(p) \\
\boldsymbol{E}T(i|p) &= n_p n_i\,\mu^2 + \; n_i s_p\sigma^2(pi) + \; n_p n_i\,\sigma^2(i) + \; n_i s_p\,\sigma^2(p) \\
\boldsymbol{E}T(pi|p) &= n_p n_i\,\mu^2 + n_p n_i\,\sigma^2(pi) + \; n_p n_i\,\sigma^2(i) + n_p n_i\,\sigma^2(p) \\
\boldsymbol{E}T(\mu|p) &= n_p n_i\,\mu^2 + \quad s_p\,\sigma^2(pi) + \quad n_p\,\sigma^2(i) + \; n_i s_p\,\sigma^2(p).
\end{aligned} \tag{49}$$

3.3 Boot–p, i

Boot–p, i involves the independent application of boot–p and boot–i. It follows that under boot–p, i, the boot–i results in Equations 31 and 33 still apply—i.e.,

$$\frac{n_p}{n_i}\,\boldsymbol{E}\left(\sum_{i=1}^{n_i}\nu_i\right)^2 = n_p s_i\,\sigma^2(i) \quad \text{and} \quad \frac{n_p}{n_i}\,\boldsymbol{E}\left(\sum_{i=1}^{n_i}\nu_{pi}\right)^2 = n_p s_i\,\sigma^2(pi).$$

Similarly, the boot–p results in Equations 46 and 47 still apply—i.e.,

$$\frac{n_i}{n_p}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\nu_p\right)^2 = n_i s_p\,\sigma^2(p) \quad \text{and} \quad \frac{n_i}{n_p}\boldsymbol{E}\left(\sum_{p=1}^{n_p}\nu_{pi}\right)^2 = n_i s_p\,\sigma^2(pi).$$

However,

$$\frac{1}{n_p n_i}\left(\sum_{p=1}^{n_p}\sum_{i=1}^{n_i}\nu_{pi}\right)^2$$

is not given by Equation 26 (no bootstrapping), nor by Equation 35 (boot–i), nor by Equation 48 (boot–p). Here it is shown that under boot–p, i,

$$\frac{1}{n_p n_i}\left(\sum_{p=1}^{n_p}\sum_{i=1}^{n_i}\nu_{pi}\right)^2 = s_p s_i\,\sigma^2(pi). \tag{50}$$

The crux of the matter is to show that

$$\left(\sum_{p=1}^{n_p}\sum_{i=1}^{n_i}\nu_{pi}\right)^2 = (2n_p - 1)(2n_i - 1)\,\sigma^2(pi). \tag{51}$$

This result is verified next using both a simplified and a detailed derivation.

3.3.1 Simplified Derivation of $(\sum_p \sum_i \nu_{pi})^2$

Consider, Table 3, which provides a matrix display of possible results for a particular replication of boot–p, i with $n_p = 3$ persons and $n_i = 4$ items. In Table 3 the entry for all non-blank cells is $\sigma^2(pi)$. These non-blank cells are instances in which $p = p'$ and $i = i'$, which will be called a "match." We will say that the cells that involve two persons (possibly the same) are "major" cells. There are nine major cells in Table 3, three of which are diagonal, and six of which are off-diagonal.

Table 3: Illustration of when $\boldsymbol{E}\nu_{pi}\nu_{p'i'} = \sigma^2(pi)$ for one Possible Replication of Boot–p, i with $n_p = 3$ and $n_i = 4$

		p_1				p_1				p_3			
		i_1	i_3	i_3	i_4	i_1	i_3	i_3	i_4	i_1	i_3	i_3	i_4
p_1	i_1	$*$				p							
	i_3		$*$	i			p	p, i					
	i_3		i	$*$			p, i	p					
	i_4				$*$				p				
p_1	i_1	p				$*$							
	i_3		p	p, i			$*$	i					
	i_3		p, i	p			i	$*$					
	i_4				p				$*$				
p_3	i_1									$*$			
	i_3										$*$	i	
	i_3										i	$*$	
	i_4												$*$

Note, in particular, that for any of the nine major cells, the items resulting from boot-i are the same, as they must be for boot–p, i. From section 3.1, it necessarily follows that for any major diagonal cell the expected number of matches is $(2n_i - 1)$. Thus, over all major diagonal cells

$$\boldsymbol{E}(\text{number of matches in major diagonal cells}) = n_p\,(2n_i - 1). \tag{52}$$

A match can occur for a major off-diagonal cell only if boot-p results in two persons being the same. From section 3.2 the expected number of times this occurs is $n_p - 1$. Whenever this occurs, the expected number of matches is $(2n_i - 1)$ (see section 3.1). It follows that

$$\boldsymbol{E}(\text{number of matches in major off-diagonal cells}) = (n_p - 1)(2n_i - 1). \tag{53}$$

Adding Equations 52 and 53 gives

$$\boldsymbol{E}(\text{number of matches}) = (2n_p - 1)(2n_i - 1). \tag{54}$$

3.3.2 Detailed Derivation of $(\sum_p \sum_i \nu_{pi})^2$

Let us reconsider Equation 15. The second term in Equation 15 can be split into three parts, giving:

$$\boldsymbol{E}\left(\sum_{p=1}^{n_p}\sum_{i=1}^{n_i}\nu_{pi}\right)^2 \quad = \quad \sum_{p\,\cap\,i}\sum \boldsymbol{E}\nu_{pi}^2$$

$$+\sum\sum_{p\cap(i\neq i')}\sum \boldsymbol{E}(\nu_{pi}\nu_{pi'})$$
$$+\sum\sum_{(p\neq p')\cap i}\sum \boldsymbol{E}(\nu_{pi}\nu_{p'i})$$
$$+\sum\sum_{(p\neq p')\cap(i\neq i')}\sum\sum \boldsymbol{E}(\nu_{pi}\nu_{p'i'}). \tag{55}$$

The expression $p \cap (i \neq i')$ means the person is the same, and the items occupy different ordinal positions, although the "pair" of items may be the same as a result of bootstrapping items. The expression $(p \neq p') \cap i$ means the item is the same, and the persons occupy different ordinal positions, although the "pair" of persons may be the same as a result of bootstrapping persons. The expression $(p \neq p') \cap (i \neq i')$ means the persons occupy different ordinal positions, the items occupy different ordinal positions, and there may be a person-item match as a result of bootstrapping both persons and items.

The first term in Equation 55 necessarily results in $n_p n_i$ repetitions of $\sigma^2(pi)$. The second term results is some number of repetitions of $\sigma^2(pi)$ depending on the particular boot–i sample. The third term results is some number of repetitions of $\sigma^2(pi)$ depending on the particular boot–p sample. The fourth term results is some number of repetitions of $\sigma^2(pi)$ depending on the particular boot–i and boot–p samples.

As an example, consider again Table 3. The symbols have the following meaning:

$*$ — necessarily equals $\sigma^2(pi)$, with or without bootstrapping;

i — equals $\sigma^2(pi)$ as a chance occurrence resulting from boot–i, only;

p — equals $\sigma^2(pi)$ as a chance occurrence resulting from boot–p, only; and

p, i — equals $\sigma^2(pi)$ as a chance occurrence resulting from both boot–p and boot–i.

To understand Equation 55 and the example in Table 3, it is crucial to recognize that under boot–p, i, when bootstrapping persons results in two persons being the same, the items are also the same. Similarly, when bootstrapping items results in two items being the same, the persons are also the same.

For each of the four possibilities discussed above, Table 4 provides

- the notation in Table 3,
- the set theory notation in Equation 55,
- the probability of a match,
- the possible number of matches, and
- the expected number of matches (the product of the previous two rows).

Note that the sum of the terms in the last row is identical to the result in Equation 54.

Table 4: Obtaining Multiplier of $\sigma^2(pi)$ for Boot–p, i

	Major Diagonal		Major Off-Diagonal	
	Sub Diagonal	Sub Off-Diagonal	Sub Diagonal	Sub Off-Diagonal
Symbol in Table 3	$*$	i	p	p, i
Notation in Equation 55	$p \cap i$	$p \cap (i \neq i')$	$(p \neq p') \cap i$	$(p \neq p') \cap (i \neq i')$
Pr(match)	1	$\dfrac{1}{n_i}$	$\dfrac{1}{n_p}$	$\dfrac{1}{n_p n_i}$
Possible Number of Matches	$n_p n_i$	$n_p[n_i(n_i-1)]$	$[n_p(n_p-1)]n_i$	$[n_p(n_p-1)][n_i(n_i-1)]$
Expected Number of Matches	$n_p n_i$	$n_p(n_i-1)$	$(n_p-1)n_i$	$(n_p-1)(n_i-1)$

3.3.3 Expected T Terms

It follows that the expected T terms under boot–p, i are:

$$\begin{aligned}
\boldsymbol{E}T(p|p) &= n_p n_i\,\mu^2 + n_p s_i \sigma^2(pi) + n_p s_i\,\sigma^2(i) + n_p n_i\,\sigma^2(p) \\
\boldsymbol{E}T(i|p) &= n_p n_i\,\mu^2 + n_i s_p\,\sigma^2(pi) + n_p n_i\,\sigma^2(i) + n_i s_p\,\sigma^2(p) \\
\boldsymbol{E}T(pi|p) &= n_p n_i\,\mu^2 + n_p n_i\,\sigma^2(pi) + n_p n_i\,\sigma^2(i) + n_p n_i\,\sigma^2(p) \\
\boldsymbol{E}T(\mu|p) &= n_p n_i\,\mu^2 + s_p s_i \sigma^2(pi) + n_p s_i\,\sigma^2(i) + n_i s_p\,\sigma^2(p).
\end{aligned} \tag{56}$$

3.4 Summary

For the $p \times i$ design, the $\boldsymbol{E}T(\alpha)$ results can be obtained quite easily, as discussed below.

3.4.1 Rule for Expected T Terms under boot–λ

Let λ be the set of facets that are bootstrapped (e.g., p, i, or both p and i), with each of them denoted λ_j. Then, the $\boldsymbol{E}T(\alpha|\lambda)$ terms for the $p \times i$ design can all be obtained using the following rule:

> *T-terms Rule.* For each λ_j consider each of the no-bootstrapping $\boldsymbol{E}T(\alpha)$ equations. If α does *not* contain λ_j, locate the variance components that *do* contain λ_j, and multiple their no-bootstrapping T-term coefficients by s_{λ_j}.

The resulting set of equations can be solved to obtain the estimated variance components, $\hat{\sigma}^2(\alpha)$. As discussed in the rest of this paper, this basic rule also applies to any multifacet balanced design.

3.4.2 Other Results Involving boot–λ

Using the expected T-term equations, it is also possible to express:

- expected boot–λ mean squares in terms of variance components,
- estimated variance components in terms of boot–λ mean squares, and
- estimated variance components in terms of estimated variance components that result directly from boot–λ (see Appendix A).

These additional sets of equations can be useful, but they are not necessary to derive estimated variance components. The expected T-term equations are sufficient for that purpose.

4 The $p \times (i\!:\!h)$ Design

To illustrate how the *T-terms Rule* in section 3.4.1 can be extended to multifacet designs we consider the $p \times (i\!:\!h)$ design. We begin by reviewing how T terms can be used to estimate variance components under the no-bootstrapping condition. Then we consider the various bootstrap procedures that might be used. For each bootstrap procedure Appendix B provides various bootstrap results including:

1. the coefficients of the variance components in the expected T terms,
2. expected boot–λ mean squares in terms of variance components, and
3. estimated variance components in terms of estimated variance components that result directly from boot–λ.

In the bootstrap discussion that begins in section 4.2, we focus only on the coefficients of the T terms. As illustrated in section 3, it is strightforward (although sometimes tedious) to derive various expressions for the estimated variance components once the expected T-term equations are obtained.

4.1 No Bootstrapping

For the $p \times (i\!:\!h)$ design, the T terms are

$$T(p) = n_i n_h \sum \overline{X}_p^2,\ T(h) = n_p n_i \sum \overline{X}_h^2,\ T(i{:}h) = n_p \sum\sum \overline{X}_{i:h}^2,$$

$$T(ph) = n_i \sum\sum \overline{X}_{ph}^2,\ T(pi{:}h) = \sum\sum\sum X_{pi:h}^2, \quad \text{and}\ \ T(\mu) = n_p n_i n_h \overline{X}^2.$$

Table 5: Coefficients of μ^2 and Variance Components in Expected Values of T Terms for Balanced $p \times (i:h)$ Design

	Coefficients					
$\boldsymbol{E}T$ term	μ^2	$\sigma^2(pi{:}h)$	$\sigma^2(ph)$	$\sigma^2(i{:}h)$	$\sigma^2(h)$	$\sigma^2(p)$
$\boldsymbol{E}T(p)$	$n_p n_i n_h$	n_p	$n_p n_i$	n_p	$n_p n_i$	$n_p n_i n_h$
$\boldsymbol{E}T(h)$	$n_p n_i n_h$	n_h	$n_i n_h$	$n_p n_h$	$n_p n_i n_h$	$n_i n_h$
$\boldsymbol{E}T(i{:}h)$	$n_p n_i n_h$	$n_i n_h$	$n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_i n_h$
$\boldsymbol{E}T(ph)$	$n_p n_i n_h$	$n_p n_h$	$n_p n_i n_h$	$n_p n_h$	$n_p n_i n_h$	$n_p n_i n_h$
$\boldsymbol{E}T(pi{:}h)$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$
$\boldsymbol{E}T(\mu)$	$n_p n_i n_h$	1	n_i	n_p	$n_p n_i$	$n_i n_h$

Table 5 provides the coefficients of the variance components for the expected values of these T terms. These coefficients can be obtained using formulas in Brennan (2001, p. 219) that are provided later in section 5.

Replacing expected T terms with actual T terms, and replacing $\sigma^2(\alpha)$ with $\hat\sigma^2(\alpha)$, gives a set of six equations in six unknowns which can be solved for unique, unbiased estimates of the variance components (as well as μ^2):

$$
\begin{aligned}
\hat\sigma^2(p) &= \frac{n_h T(p) + T(h) - T(ph) - n_h T(\mu)}{n_i n_h (n_p - 1)(n_h - 1)} \\
\hat\sigma^2(h) &= \frac{\begin{bmatrix} n_h(n_i-1)T(p) \\ +[n_h^2(n_i-1) + n_p(n_h-1)]T(h) - n_p(n_h-1)T(i{:}h) \\ -(n_h n_i - 1)T(ph) + (n_h-1)T(pi{:}h) - n_h^2(n_i-1)T(\mu) \end{bmatrix}}{n_p n_i n_h (n_p-1)(n_i-1)(n_h-1)} \\
\hat\sigma^2(i{:}h) &= \frac{-n_p T(h) + n_p T(i{:}h) + T(ph) - T(pi{:}h)}{n_p n_h (n_p-1)(n_i-1)} \\
\hat\sigma^2(ph) &= \frac{\begin{bmatrix} -n_h(n_i-1)T(p) - (n_i n_h - 1)T(h) + (n_h-1)T(i{:}h) \\ +(n_i n_h - 1)T(ph) - (n_h-1)T(pi{:}h) + n_h(n_i-1)T(\mu) \end{bmatrix}}{n_i n_h (n_p-1)(n_i-1)(n_h-1)} \\
\hat\sigma^2(pi{:}h) &= \frac{T(pi{:}h) - T(ph) - T(i{:}h) + T(h)}{n_h(n_p-1)(n_i-1)}.
\end{aligned}
\tag{57}
$$

Although the results in Equation Set 57 are complex, they are easily obtained using matrix procedures. In this sense, mathematically, using T terms is a very direct way of obtaining unbiased estimates of the variance components. Traditionally, however, for computational ease with a hand calculator the usual procedure is to obtain mean squares and expected mean squares, and then use these to obtain the estimated variance components. Mean squares with respect to T terms for the $p \times (i:h)$ design are:

$$MS(p) = \frac{T(p) - T(\mu)}{n_p - 1}$$

$$
\begin{aligned}
MS(h) &= \frac{T(h) - T(\mu)}{n_h - 1} \\
MS(i{:}h) &= \frac{T(i{:}h) - T(h)}{n_h(n_i - 1)} \\
MS(ph) &= \frac{T(ph) - T(p) - T(h) + T(\mu)}{(n_p - 1)(n_h - 1)} \\
MS(pi{:}h) &= \frac{T(pi{:}h) - T(ph) - T(i{:}h) + T(h)}{n_h(n_p - 1)(n_i - 1)}.
\end{aligned} \tag{58}
$$

The expected values of the mean squares are:

$$
\begin{aligned}
\boldsymbol{E}MS(p) &= \sigma^2(pi{:}h) + n_i\,\sigma^2(ph) + n_i n_h\,\sigma^2(p) \\
\boldsymbol{E}MS(h) &= \sigma^2(pi{:}h) + n_i\,\sigma^2(ph) + n_p\,\sigma^2(i{:}h) + n_i n_p\,\sigma^2(h) \\
\boldsymbol{E}MS(i{:}h) &= \sigma^2(pi{:}h) + n_p\,\sigma^2(i{:}h) \\
\boldsymbol{E}MS(ph) &= \sigma^2(pi{:}h) + n_i\,\sigma^2(ph) \\
\boldsymbol{E}MS(pi{:}h) &= \sigma^2(pi{:}h).
\end{aligned} \tag{59}
$$

When expected mean squares are replaced with actual mean squares, the resulting equations can be solved in several ways for the estimated variance components. Doing so gives the following:

$$
\begin{aligned}
\sigma^2(p) &= \frac{MS(p) - MS(ph)}{n_i n_h} \\
\hat{\sigma}^2(h) &= \frac{MS(h) - MS(i{:}h) - MS(ph) + MS(pi{:}h)}{n_p n_i} \\
\hat{\sigma}^2(i{:}h) &= \frac{MS(i{:}h) - MS(pi{:}h)}{n_p} \\
\hat{\sigma}^2(ph) &= \frac{MS(ph) - MS(pi{:}h)}{n_i} \\
\hat{\sigma}^2(pi{:}h) &= MS(pi{:}h)
\end{aligned} \tag{60}
$$

These results are identical to those in Equation Set 57 that were obtained by solving the T-term equations directly for estimated variance components.

4.2 Boot–p, Boot–i, and Boot–h

Section 3.4.1 provided a rule for obtaining coefficients of T terms under boot–p and boot–i for the $p \times i$ design. This rule has broader applicability, however. It applies when bootstrapping any single facet, no matter how many of them there may be in a design. For the $p \times (i{:}h)$ design, this means that this rules applies to boot–p, boot–i, and boot–h. The rule is repeated here for ease of reference.

> *T-terms Rule*: For each λ_j consider each of the no-bootstrapping $\boldsymbol{E}T(\alpha)$ equations. If α does *not* contain λ_j, locate the variance components that *do* contain λ_j, and multiple their no-bootstrapping T-term coefficients by s_{λ_j}.

For the $p \times (i\!:\!h)$ design, the expected T terms and other results for boot–p, boot–i, and boot–h are provided in Appendix B.

The fact that this rule applies for bootstrapping any single facet is evident from the form of the derivation in section 3.1 for boot–i for the $p \times i$ design. Consider, for example, $\boldsymbol{E}T(p) = n_i n_h \boldsymbol{E} \sum \overline{X}_p^2$ for the $p \times (i\!:\!h)$ design. Suppose we focus on boot–h and the ν_h term. Clearly, the bootstrap facet h is not in μ, but h is in ν_h, which means that the conditions for the *T-terms Rule* apply. Since $T(p)$ involves the square of average effects over i and h, it follows that the ν_h term in $\boldsymbol{E}T(p)$ is

$$\begin{aligned} n_i n_h \boldsymbol{E}\left[\sum_p \left(\frac{\sum_h \sum_i \nu_h}{n_i n_h}\right)^2\right] &= \frac{n_p}{n_i n_h}\boldsymbol{E}\left(\sum_h \sum_i \nu_h\right)^2 \\ &= \frac{n_p}{n_i n_h}\boldsymbol{E}\left(n_i \sum_h \nu_h\right)^2 \\ &= \frac{n_p n_i}{n_h}\boldsymbol{E}\left(\sum_h \nu_h\right)^2. \end{aligned}$$

The form of this result mirrors that of Equation 31 for the ν_i effect in $T(p)$ under boot–i in the $p \times i$ design. It follows that

$$\frac{n_p n_i}{n_h}\boldsymbol{E}\left(\sum_h \nu_h\right)^2 = \left(\frac{n_p n_i}{n_h}\right)(2n_h - 1)\sigma^2(h) = n_p n_i s_h \sigma^2(h).$$

This type of result necessarily occurs whenever λ_j (h in this case) is not in α (p in this case), and the effect under consideration (ν_h in this case) involves λ_j (h in this case).

Note that the above *T-terms Rule* for boot–λ_j applies to boot–i in the $p \times (i\!:\!h)$ design even though there is no i effect in that design. Although i is nested within h, it is still possible to bootstrap i, only.

4.3 Boot–p, h and Boot–p, i

When a pair of facets is bootstrapped, the *T-terms Rule* in section 4.2 can be applied for both of them independently, as illustrated in section 3.3 for boot–p, i with the $p \times i$ design. The fact that the derivation in section 3.3 was couched in the context of the $p \times i$ design is *not* restrictive.

Consider, for example, $\boldsymbol{E}T(\mu) = n_p n_i n_h \boldsymbol{E}\overline{X}^2$ for the $p \times (i\!:\!h)$ design. Suppose we focus on boot–p, h and the $\nu_{pi:h}$ term. Clearly, the bootstrap facets (p and h) are not in μ, but they are in $\nu_{pi:h}$, which means that the *T-terms Rule* applies. Since $T(\mu)$ involves the square of average effects over all three facets, it follows that the $\nu_{pi:h}$ term in $\boldsymbol{E}T(\mu)$ is

$$n_p n_i n_h \boldsymbol{E}\left(\frac{\sum_p \sum_h \sum_i \nu_{pi:h}}{n_p n_i n_h}\right)^2 = \frac{1}{n_p n_i n_h}\boldsymbol{E}\left(\sum_p \sum_h \sum_i \nu_{pi:h}\right)^2$$

$$= \frac{1}{n_p n_h}\boldsymbol{E}\left(\sum_p \sum_h \nu_{pi:h}\right)^2, \tag{61}$$

since, when only p and h are being bootstrapped,

$$\boldsymbol{E}\left(\sum_p \sum_h \sum_i \nu_{pi:h}\right)^2 = n_i \boldsymbol{E}\left(\sum_p \sum_h \nu_{pi:h}\right)^2.$$

The form of the result in Equation 61 mirrors that of Equation 50 for the ν_{pi} effect in $T(\mu)$ under boot–p, i in the $p \times i$ design. It follows that

$$\frac{1}{n_p n_h}\boldsymbol{E}\left(\sum_p \sum_h \nu_{pi:h}\right)^2 = s_p s_h \sigma^2(pi{:}h).$$

This type of result necessarily occurs whenever neither of the indices in λ (p and h in this case) are in α (μ in this case), and the effect under consideration ($\nu_{pi:h}$ in this case) involves both of the indices in λ (p and h in this case).

The type of logic discussed above for boot–p, h also applies to boot–p, i in the $p \times (i{:}h)$ design even though there is no pi effect in the design. For the $p \times (i{:}h)$ design, the expected T terms and other results for boot–p, h and boot–p, i are provided in Appendix B.

In short, we have demonstrated that the *T-terms Rule* in section 4.2 applies for bootstrapping single facets and pairs of non-nested facets. Next we show that the *T-terms Rule* also applies to nested bootstrap facets. Subsequently, we show that it applies when more than two facets are bootstrapped.

4.4 Boot–i, h with i Nested within h

Boot–i, h means bootstrap i and independently bootstrap h. Clearly, we can bootstrap i within each level of h, and then we can bootstrap h. In this section it is demonstrated that when boot–i, h is conducted in this manner, the *T-terms Rule* in section 4.2 gives the correct results. That is, the fact that the facets are nested has no bearing on the coefficients of the T terms. The crux of the matter is to show that under boot–i, h the coefficients of $\sigma^2(i{:}h)$ and $\sigma^2(pi{:}h)$ in $\boldsymbol{E}T(p)$ and $\boldsymbol{E}T(\mu)$ involve $s_i s_h$.

Let us begin with $\boldsymbol{E}T(p)$ and $\nu_{i:h}$. $\boldsymbol{E}T(p) = n_i n_h \boldsymbol{E}\sum \overline{X}_p^2$ involves the term

$$n_i n_h \boldsymbol{E}\left[\sum_p \left(\frac{\sum_h \sum_i \nu_{i:h}}{n_i n_h}\right)^2\right] = \frac{n_p}{n_i n_h}\boldsymbol{E}\left(\sum_h \sum_i \nu_{i:h}\right)^2.$$

Focus on the kernal $\boldsymbol{E}(\sum\sum \nu_{i:h})^2$. Consider Table 6 in which $n_h = 3$ and $n_i = 4$. The layout of the table mirrors that of Table 3 except that the set of items for the third level of h is not the same as that for the first level of h. [For the $p \times (i{:}h)$ design, the items are different for each unique level of h, regardless

Table 6: Illustration of when $\boldsymbol{E}\nu_{i:h}\nu_{i':h'} = \sigma^2(i{:}h)$ for one Possible Replication of Boot–i, h with $n_h = 3$, $n_i = 4$, and Boot–i Performed First

		h_1				h_1				h_3			
		i_1	i_3	i_3	i_4	i_1	i_3	i_3	i_4	i_1	i_1	i_3	i_3
h_1	i_1	$*$				h							
	i_3		$*$	i			h	i, h					
	i_3		i	$*$			i, h	h					
	i_4				$*$				h				
h_1	i_1	h				$*$							
	i_3		h	i, h			$*$	i					
	i_3		i, h	h			i	$*$					
	i_4				h				$*$				
h_3	i_1									$*$	i		
	i_1									i	$*$		
	i_3											$*$	i
	i_3											i	$*$

of whether or not bootstrapping is performed.] Note, in particular, that the set of items is the *same* whenever the levels of h are the same. This is precisely what happens when i is bootstrapped first.

Under these circumstances, the logic involving bootstrapping two facets (see section 3.3) still applies, and under boot–i, h

$$\boldsymbol{E}\left(\sum_h \sum_i \nu_{i:h}\right)^2 = (2n_i - 1)(2n_h - 1)\,\sigma^2(i{:}h). \tag{62}$$

(Note that if h were bootstrapped first, when two instances of h were the same, the set of items could be different and Equation 62 would not apply.[3]) Given the result in Equation 62, it is evident that

$$\frac{n_p}{n_i n_h}\boldsymbol{E}\left(\sum_h \sum_i \nu_{i:h}\right)^2 = n_p s_i s_h\,\sigma^2(i{:}h), \tag{63}$$

where n_p is the coefficient without bootstrapping (see Table 5), and $s_i s_h$ corrects for the dependencies induced by bootstrapping of both i and h (assuming i is bootstrapped first).

Similarly, $\boldsymbol{E}T(\mu) = n_p n_i n_h \boldsymbol{E}\overline{X}^2$ involves the term

$$n_p n_i n_h \boldsymbol{E}\left(\frac{\sum_p \sum_h \sum_i \nu_{i:h}}{n_p n_i n_h}\right)^2 \quad = \quad \frac{n_p}{n_i n_h}\boldsymbol{E}\left(\sum_h \sum_i \nu_{i:h}\right)^2$$

[3] If h is bootstrapped first, $(\sum_h \sum_i \nu_{i:h})^2$ can be determined, but the result is much more complicated than Equation 62.

Table 7: Obtaining Multiplier of $\sigma^2(pih)$ for Boot–p, i, h

	h_1	h_3	h_3
h_1	Table 3	Table 3	Table 3
h_3	Table 3	Table 3	Table 3
h_3	Table 3	Table 3	Table 3

$$= n_p s_i s_h \, \sigma^2(i{:}h).$$

Using the same logic, it can be shown that under boot–i, h the coefficients of $\sigma^2(pi{:}h)$ in $\boldsymbol{E}T(p)$ and $\boldsymbol{E}T(\mu)$ involve $s_i s_h$.

4.5 Boot–p, i, h

Strictly speaking, it has not yet been demonstrated that the *T-terms Rule* yields the correct result for the simultaneous application of all three bootstrap procedures to the $\nu_{pi:h}$ term in $\boldsymbol{E}T(\mu) = n_p n_i n_h \boldsymbol{E}\overline{X}^2$. [See Appendix B for the expected T terms and other results under boot–p, i, h.] That is, we need to show that

$$\begin{aligned} n_p n_i n_h \boldsymbol{E}\left(\frac{\sum_p \sum_i \sum_h \nu_{pi:h}}{n_p n_i n_h}\right)^2 &= \frac{1}{n_p n_i n_h}\boldsymbol{E}\left(\sum_p \sum_i \sum_h \nu_{pi:h}\right)^2 \\ &= s_p s_i s_h \sigma^2(pi{:}h). \end{aligned}$$

To demonstrate this result, it is sufficient to show that

$$\left(\sum_p \sum_i \sum_h \nu_{pi:h}\right)^2 = (2n_h - 1)(2n_p - 1)(2n_i - 1)\sigma^2(pi{:}h).$$

Suppose $n_p = 3$, $n_i = 4$, $n_h = 3$, and a particular replication of boot–p, i, h results in $(p_1,\, p_1,\, p_3)$, $(i_1,\, i_3,\, i_3,\, i_4)$, and $(h_1,\, h_3,\, h_3)$. This is represented in Table 7, where the reference to Table 3 in each of the nine major cells means that Table 3 is replicated in each of these cells. We know from section 3.3 that for each of the major diagonal cells in Table 7 the expected number of matches is $(2n_p - 1)(2n_i - 1)$. Thus, over all major diagonal cells

$$\boldsymbol{E}(\text{number of matches in major diagonal cells}) = n_h\,[(2n_p - 1)(2n_i - 1)]. \quad (64)$$

A match can occur for a major off-diagonal cell in Table 7 only if boot–h results in two levels of h being the same. By analogy with the discussion in section 3.2, the expected number of times this occurs is $n_h - 1$. Whenever this occurs, the expected number of matches is $(2n_p - 1)(2n_i - 1)$. It follows that

$$\boldsymbol{E}(\text{number of matches in major off-diagonal cells}) = (n_h - 1)[(2n_p - 1)(2n_i - 1)]. \quad (65)$$

Adding Equations 64 and 65 gives

$$\boldsymbol{E}(\text{number of matches}) = (2n_h - 1)(2n_p - 1)(2n_i - 1), \quad (66)$$

which implies that

$$\left(\sum_p \sum_i \sum_h \nu_{pi:h}\right)^2 = (2n_h - 1)(2n_p - 1)(2n_i - 1)\sigma^2(pi{:}h).$$

5 General Procedures for any Balanced Design

The $p \times (i{:}h)$ design provides a microcosm of the complexities that can arise in the types of designs encountered in generalizability theory. Consequently, extending the boot–λ results reported in section 4 to other designs is relatively straightforward. This section provides general procedures and results for any design and any bootstrap procedure.

5.1 $\boldsymbol{T}$ Terms

For any component α, the T term for a balanced design (see Brennan, 2001, p. 217) is

$$T(\alpha) = \left[\prod n(\sim \alpha)\right] \sum_\alpha \overline{X}^2_\alpha, \quad (67)$$

where $\prod n(\sim \alpha)$ is the product of the sample sizes for the indexes *not* in α, and the summation is over all of the indexes in α.[4]

5.2 Expected $\boldsymbol{T}$ terms for Balanced Designs with no Bootstrapping

Brennan (2001, p. 219, Equation 7.3) provides an equation for obtaining no-bootstrapping expected T terms that applies for both balanced and unbalanced

[4]For consistency with notational conventions used later in this paper, Equation 67 is stated in a manner slightly different from that in Brennan (2001, p. 217).

designs. For balanced-designs only, the equation is somewhat simpler. Specifically, the coefficient of $\sigma^2(\alpha)$ in the expected value of the T term for β is

$$k[\sigma^2(\alpha), \boldsymbol{E}T(\beta)] = \prod n(\beta) \prod n[\sim (\alpha \cap \beta)], \tag{68}$$

where $\prod n(\beta)$ is to be interpreted as the product of the sample sizes for all indices in β and $\prod n[\sim (\alpha \cap \beta)]$ means the product of the sample sizes for all indices that are not in either α or β.

5.3 Expected T terms Under boot–λ

For any design and any bootstrap procedure, expected T terms are relatively easy to obtain, starting with the expected T-term equations for a balanced-design without any bootstrapping. Let λ be the set of indexes for the facets that are bootstrapped, and let λ_j be any one of the m indexes in λ. Define

$$s_{\lambda_j} = \frac{2n_{\lambda_j} - 1}{n_{\lambda_j}}. \tag{69}$$

5.3.1 Rule

To obtain the coefficients of each of the variance components in the expected T terms for the boot–λ procedure, apply the following rule to each of the coefficients for the non-bootstrapped balanced-design given by Equation 68:

> *T-terms Rule*: For each λ_j consider each of the no-bootstrapping $\boldsymbol{E}T(\alpha)$ equations. If α does *not* contain λ_j, locate the variance components that *do* contain λ_j, and multiple their no-bootstrapping T-term coefficients by s_{λ_j}.

5.3.2 Algorithm

Let each of the expected T terms under boot–λ be designated $\boldsymbol{E}T(\beta|\gamma)$ with the variance components designated as $\sigma^2(\alpha)$. Then, the above rule can also be expressed in terms of the following algorithm:

Algorithm 1

1. multiply the no-bootstrapping T-term coefficient by s_{λ_1} if $\lambda_1 \in \alpha \;\cap\; \lambda_1 \notin \beta$
2. multiply the coefficient in the previous step by s_{λ_2} if $\lambda_2 \in \alpha \;\cap\; \lambda_2 \notin \beta$

$\vdots$

m. multiply the coefficient in the previous step by s_{λ_m} if $\lambda_k \in \alpha \cap \lambda_k \notin \beta$.

For those unfamiliar with set theory notation, $\in$ means "in," $\notin$ means "not in," and $\cap$ means "and."

5.3.3 General Equation for $\boldsymbol{T}$ Terms under Boot–$\boldsymbol{\lambda}$

The result of applying the above rule or algorithm can be expressed in a single equation as:

$$k[\sigma^2(\alpha), \boldsymbol{E}T(\beta|\lambda)] = \prod n(\beta) \prod n[\sim (\alpha \cap \beta)] \prod [s_{\lambda_j} | \lambda_j \in \alpha \ \cap \ \lambda_j \notin \beta], \quad (70)$$

where the last term is the product of s_{λ_j} for all λ_j such that $\lambda_j \in \alpha \ \cap \ \lambda_j \notin \beta$.

Replacing the left side of the resulting expected T-term equations for boot–λ with actual T terms, and replacing all $\sigma^2(\alpha)$ with $\hat{\sigma}^2(\alpha)$, the resulting estimation equations can be solved for the $\hat{\sigma}^2(\alpha)$, which are the so called "ANOVA" unbiased estimates of the variance components. Computationally, this is the usually the most direct way to estimate the variance components. However, researchers sometimes want to solve the expected mean square equations under boot–λ for the estimated variance components, and/or researchers may want the $\hat{\sigma}^2(\alpha)$ expressed in terms of the bootstrap estimates, $\hat{\sigma}^2(\alpha|\lambda)$.

5.4 Expected Mean Squares Under boot–$\boldsymbol{\lambda}$

There are at least two procedures that can be used to obtain the expected mean square equations under the boot–λ procedure, $\boldsymbol{E}MS(\beta|\lambda)$. (Note that the right side of these equations are expressions in terms of the actual variance components, not the bootstrap versions of them.) By definition, these expected mean square equations can be obtained by applying the expected T terms for boot–λ with the mean squares with respect to boot–λ T terms (see, for example, Equation Set 39). The algebra required to do so can be tedious, however. An algorithm described below provides an alternative, simpler procedure.

First, for a random model with a balanced design and no bootstrapping, the following is a general expression for the expected mean square equations (see Brennan, 2001, p. 77):

$$\boldsymbol{E}MS(\beta) = \sum \left\{ \left[\prod n(\sim \alpha) \right] \sigma^2(\alpha) \right\}, \quad (71)$$

where the summation is taken over all α that contain *at least all of the indexes* in β, and $\prod n(\sim \alpha)$ is the product of the sample sizes for the indexes not in α. See, for example, Equation Set 59.

5.4.1 Algorithm

Now, let

$$t_{\lambda_j} = \frac{n_{\lambda_j} - 1}{n_{\lambda_j}}, \quad (72)$$

and let β^* be the set of primary indexes (those prior to any colon) in β. To obtain the coefficients of each of the variance components in the expected mean squares for the boot–λ procedure, $\boldsymbol{E}MS(\beta|\lambda)$, apply the following algorithm to each of the coefficients for the non-bootstrapped balanced design:

Algorithm 2

1. provided λ_1 is not a nesting index[5] in *both* α and β, multiply the coefficient by

   t_{λ_1} if $\lambda_1 \in \alpha \ \cap \ \lambda_1 \in \beta^*$ or by s_{λ_1} if $\lambda_1 \in \alpha \ \cap \ \lambda_1 \notin \beta^*$

2. provided λ_2 is not a nesting index in *both* α and β, multiply the coefficient in the previous step by

   t_{λ_2} if $\lambda_2 \in \alpha \ \cap \ \lambda_2 \in \beta^*$ or by s_{λ_2} if $\lambda_2 \in \alpha \ \cap \ \lambda_2 \notin \beta^*$

$\vdots$

k. provided λ_k is not a nesting index in *both* α and β, multiply the coefficient in the previous step by

   t_{λ_k} if $\lambda_k \in \alpha \ \cap \ \lambda_k \in \beta^*$ or by s_{λ_k} if $\lambda_k \in \alpha \ \cap \ \lambda_k \notin \beta^*$.

5.4.2 Rule

The above algorithm gives the same results as applying the following rule to each of the coefficients in the $\boldsymbol{EMS}(\beta)$ equations:

> $\boldsymbol{E}$*MS Rule*. Provided λ_j is not a nesting index (i.e., any index after a colon) in *both* α and β, multiply the coefficient by t_{λ_1} if $\lambda_1 \in \alpha \ \cap \ \lambda_1 \in \beta^*$, or by s_{λ_1} if $\lambda_1 \in \alpha \ \cap \ \lambda_1 \notin \beta^*$.

5.5 Bias-Corrected Estimates of Variance Components

It is possible to use the $\boldsymbol{EMS}(\beta|\lambda)$ equations to solve for unbiased estimates of the variance components, $\hat{\sigma}^2(\alpha)$, in terms of the boot–λ estimates, $\hat{\sigma}^2(\alpha|\lambda)$. The resulting equations are sometimes called the bias-corrected estimates of variance components, where it is understood that the bias is induced by boot–λ. In general, for nested designs the resulting expressions for $\hat{\sigma}^2(\alpha)$ in terms of boot–λ estimates do not seem to have any reasonably simple predictable form, as indicated by the results in Appendix B for the $p \times (i\!:\!h)$ design. For fully crossed designs, however, there is a regularity to the resulting equations, as illustrated in Appendix C for the $p \times i \times h$ design.

5.6 Crossed Designs

For a fully crossed design, at least two approaches can be used for directly obtaining an unbiased estimate of $\sigma^2(\alpha)$ in terms of boot–λ estimates. One approach is an algorithm; the other is a very complicated equation.

[5]In the conventions used in Brennan (2001), a nesting index is any index after a colon.

5.6.1 Algorithm

Let

- κ be the set of indices in *both* α and λ, with each one denoted κ_j ($j = 1, \ldots, y$); and
- ζ be the set of indices in λ that are *not* in α, with each one denoted ζ_g ($g = 1, \ldots, z$).

Also, let

$$\hat{\sigma}^2(\alpha) = \frac{1}{\prod_j^y t_{\kappa_j}} \left[\begin{array}{c} \text{some function} \\ \text{of } \hat{\sigma}^2(\beta|\lambda) \end{array} \right], \tag{73}$$

with the appropriate function of the $\hat{\sigma}^2(\beta|\lambda)$ obtained using the following algorithm.

Algorithm 3

Step 0: $\hat{\sigma}^2(\alpha|\lambda)$;

Step 1: *Minus* the $\hat{\sigma}^2(\beta|\lambda)$ for all β that consist of the indexes in α and exactly *one* of the indexes in ζ, with each such $\hat{\sigma}^2(\beta|\lambda)$ divided by the degrees of freedom for the one additional index;

Step 2: *Plus* the $\hat{\sigma}^2(\beta|\lambda)$ for all β that consist of the indexes in α and any *two* of the indexes in ζ, with each such $\hat{\sigma}^2(\beta|\lambda)$ divided by the degrees of freedom for the two additional indexes;

$\vdots$

Step z: *Plus* (if z is even) or *Minus* (if z is odd) the $\hat{\sigma}^2(\beta|\lambda)$ that consists of the indexes in α and all z of the indexes in ζ, with $\hat{\sigma}^2(\beta|\lambda)$ divided by the degrees of freedom for the z indexes.

The fraction before the braces in Equation 73 is 1 if κ is the null set (i.e., contains no indexes). When ζ consists of $z = 0$ indexes, the algorithm terminates at Step 0; otherwise, the algorithm terminates at Step z with one term added or subtracted. The results reported in Appendix C for the $p \times i \times h$ design can be obtained using this algorithm, which is one way to become familiar with how the algorithm works.

5.6.2 General Equation

The above algorithm can be encapsulated in a single (very complicated) formula:

$$\sigma^2(\alpha) = \frac{1}{\prod_j^y t_{\kappa_j}} \left\{ \hat{\sigma}^2(\alpha|\lambda) + \sum_{g=1}^{z} \left[(-1)^g \sum_c \frac{\hat{\sigma}^2(\alpha\zeta^{[gc]}|\lambda)}{df(\zeta^{[gc]})} \right] \right\}, \tag{74}$$

where $\zeta^{[gc]}$ denotes a combination of g indexes in ζ, and c counts the number of such combinations—i.e.,

$$c = 1, \ldots, \binom{z}{g}.$$

5.7 Some Comments About Nested Designs

Nested designs almost always involve complexities beyond those in crossed designs. That is certainly true in the context of bootstrapping. In particular, the following matters should be kept in mind for nested designs.

- When one or more facets is/are nested within one or more other facets (e.g., $i{:}h$), the facets should be bootstrapped in the order they appear in the notational representation of effects (using the conventions in Brennan, 2001) for Equations 69–72 to be accurate.

- There are always more possible bootstrap procedures then there are effects or variance components. For example, in the $p \times (i{:}h)$ design there are five effects (p, h, $i{:}h$, ph, and $pi{:}h$), but there are seven possible bootstrap procedures (p; i; h; p, i; p, h; i, h; and p, i, h).

6 Final Comment

Probably the single most important result in this paper is the *T-terms Rule* (or its companion Equation 70), which provides the basis for specifying the expected T-term equations for any design under any bootstrap procedure. These equations can be readily solved for unbiased estimates of the random effects variance components.

7 References

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A Wiley's (2000) Bias-Corrected Estimates of Variance Components for the $p \times i$ Design

Wiley (2000) derived a complete set of bias-corrected estimates of variance components the single-facet crossed design, $p \times i$. These are listed below using the notation $\hat{\sigma}^2(\alpha)$ to designate an unbiased estimate of the variance component for α and $\hat{\sigma}^2(\alpha|\lambda)$ to designate the boot–λ estimate.

A.1 Boot-p

$$\begin{aligned}
\hat{\sigma}^2(p) &= \frac{\hat{\sigma}^2(p\,|p)}{t_p} \\
\hat{\sigma}^2(i) &= \hat{\sigma}^2(i|p) - \frac{\hat{\sigma}^2(pi|p)}{n_p - 1} \\
\hat{\sigma}^2(pi) &= \frac{\hat{\sigma}^2(pi|p)}{t_p}
\end{aligned}$$

A.2 Boot-i

$$\begin{aligned}
\hat{\sigma}^2(p) &= \hat{\sigma}^2(p|i) - \frac{\hat{\sigma}^2(pi|i)}{n_i - 1} \\
\hat{\sigma}^2(i) &= \frac{\hat{\sigma}^2(i|i)}{t_i} \\
\hat{\sigma}^2(pi) &= \frac{\hat{\sigma}^2(pi|i)}{t_i}
\end{aligned}$$

A.3 Boot-p, i

$$\begin{aligned}
\hat{\sigma}^2(p) &= \frac{1}{t_p}\left[\hat{\sigma}^2(p|p,i) - \frac{\hat{\sigma}^2(pi|p,i)}{n_i - 1}\right] \\
\hat{\sigma}^2(i) &= \frac{1}{t_i}\left[\hat{\sigma}^2(i|p,i) - \frac{\hat{\sigma}^2(pi|p,i)}{n_p - 1}\right] \\
\hat{\sigma}^2(pi) &= \frac{\hat{\sigma}^2(pi|p,i)}{t_p t_i}
\end{aligned}$$

B Bias-Corrected Estimates of Variance Components for the $p \times (i\!:\!h)$ Design

For the unbalanced $p \times (i\!:\!h)$ design and the various possible bootstrap procedures (denoted generically as boot–λ), the remaining sections of this appendix provide:

- $\boldsymbol{E}T(\beta|\lambda)$, namely, the expected T-term equations under boot–λ;
- $\boldsymbol{E}MS(\beta|\lambda)$, namely, the expected mean squares under boot–λ; and
- $\hat{\sigma}^2(\alpha)$ in terms of $\hat{\sigma}^2(\alpha|\lambda)$, namely, equations for estimating the variance components in terms of the boot–λ estimates of variance components.

Note that the third set of results are equations for obtaining unbiased estimates of the variance components, $\hat{\sigma}^2(\alpha)$, in terms of the biased boot–λ estimates of variance components. These equations are sometimes called the "bias-corrected" estimates of variance components in the sense that they correct for the bias that is induced by the bootstrap procedure. Of course, for any given instance of the bootstrap procedure, solving the T-term equations for the estimated variance components would give identically the same results. Indeed, that is usually the most computationally efficient way to proceed.

B.1 Boot–p

Table 8: Coefficients of μ^2 and Variance Components in Expected Values of T Terms for Balanced $p \times (i\!:\!h)$ Design with Boot–p

	Coefficients					
$\boldsymbol{E}T$ term	μ^2	$\sigma^2(pi{:}h)$	$\sigma^2(ph)$	$\sigma^2(i{:}h)$	$\sigma^2(h)$	$\sigma^2(p)$
$\boldsymbol{E}T(p\,\|p)$	$n_p n_i n_h$	n_p	$n_p n_i$	n_p	$n_p n_i$	$n_p n_i n_h$
$\boldsymbol{E}T(h\|p)$	$n_p n_i n_h$	$n_h s_p$	$n_i n_h s_p$	$n_p n_h$	$n_p n_i n_h$	$n_i n_h s_p$
$\boldsymbol{E}T(i{:}h\|p)$	$n_p n_i n_h$	$n_i n_h s_p$	$n_i n_h s_p$	$n_p n_i n_h$	$n_p n_i n_h$	$n_i n_h s_p$
$\boldsymbol{E}T(ph\|p)$	$n_p n_i n_h$	$n_p n_h$	$n_p n_i n_h$	$n_p n_h$	$n_p n_i n_h$	$n_p n_i n_h$
$\boldsymbol{E}T(pi{:}h\|p)$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$
$\boldsymbol{E}T(\mu\|p)$	$n_p n_i n_h$	s_p	$n_i s_p$	n_p	$n_p n_i$	$n_i n_h s_p$

Expected Mean Square Equations

$$
\begin{aligned}
\boldsymbol{E}MS(p\,|p) &= t_p\,\sigma^2(pi\!:\!h) + t_p n_i\,\sigma^2(ph) + t_p n_i n_h\,\sigma^2(p)\\
\boldsymbol{E}MS(h|p) &= s_p\,\sigma^2(pi\!:\!h) + s_p n_i\,\sigma^2(ph) + n_p\,\sigma^2(i\!:\!h) + n_p n_i\,\sigma^2(h)\\
\boldsymbol{E}MS(i{:}h|p) &= s_p\,\sigma^2(pi\!:\!h) + n_p\,\sigma^2(i\!:\!h)\\
\boldsymbol{E}MS(ph|p) &= t_p\,\sigma^2(pi\!:\!h) + t_p n_i\,\sigma^2(ph)\\
\boldsymbol{E}MS(pi{:}h|p) &= t_p\,\sigma^2(pi\!:\!h)
\end{aligned}
$$

Unbiased Estimators of Variance Components in terms of Bootstrap Estimators

$$
\begin{aligned}
\hat\sigma^2(p) &= \frac{\hat\sigma^2(p\,|p)}{t_p}\\
\hat\sigma^2(h) &= \hat\sigma^2(h|p) - \frac{\hat\sigma^2(ph|p)}{n_p - 1}\\
\hat\sigma^2(i\!:\!h) &= \hat\sigma^2(i\!:\!h|p) - \frac{\hat\sigma^2(pi\!:\!h|p)}{n_p - 1}\\
\hat\sigma^2(ph) &= \frac{\hat\sigma^2(ph|p)}{t_p}\\
\hat\sigma^2(pi\!:\!h) &= \frac{\hat\sigma^2(pi\!:\!h|p)}{t_p}
\end{aligned}
$$

B.2 Boot–*i*

Table 9: Coefficients of μ^2 and Variance Components in Expected Values of T Terms for Balanced $p \times (i\!:\!h)$ Design with Boot–i

	Coefficients					
$\boldsymbol{E}T$ term	μ^2	$\sigma^2(pi{:}h)$	$\sigma^2(ph)$	$\sigma^2(i{:}h)$	$\sigma^2(h)$	$\sigma^2(p)$
$\boldsymbol{E}T(p\vert i)$	$n_p n_i n_h$	$n_p s_i$	$n_p n_i$	$n_p s_i$	$n_p n_i$	$n_p n_i n_h$
$\boldsymbol{E}T(h\vert i)$	$n_p n_i n_h$	$n_h s_i$	$n_i n_h$	$n_p n_h s_i$	$n_p n_i n_h$	$n_i n_h$
$\boldsymbol{E}T(i{:}h\vert i)$	$n_p n_i n_h$	$n_i n_h$	$n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_i n_h$
$\boldsymbol{E}T(ph\vert i)$	$n_p n_i n_h$	$n_p n_h s_i$	$n_p n_i n_h$	$n_p n_h s_i$	$n_p n_i n_h$	$n_p n_i n_h$
$\boldsymbol{E}T(pi{:}h\vert i)$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$
$\boldsymbol{E}T(\mu\vert i)$	$n_p n_i n_h$	s_i	n_i	$n_p s_i$	$n_p n_i$	$n_i n_h$

Expected Mean Square Equations

$$
\begin{aligned}
\boldsymbol{E}MS(p|i) &= s_i\,\sigma^2(pi\!:\!h) + n_i\,\sigma^2(ph) + n_i n_h\,\sigma^2(p)\\
\boldsymbol{E}MS(h|i) &= s_i\,\sigma^2(pi\!:\!h) + n_i\,\sigma^2(ph) + s_i n_p\,\sigma^2(i\!:\!h) + n_p n_i\,\sigma^2(h)\\
\boldsymbol{E}MS(i{:}h|i) &= t_i\,\sigma^2(pi\!:\!h) + t_i n_p\,\sigma^2(i\!:\!h)\\
\boldsymbol{E}MS(ph|i) &= s_i\,\sigma^2(pi\!:\!h) + n_i\,\sigma^2(ph)\\
\boldsymbol{E}MS(pi{:}h|i) &= t_i\,\sigma^2(pi\!:\!h)
\end{aligned}
$$

Unbiased Estimators of Variance Components in terms of Bootstrap Estimators

$$
\begin{aligned}
\hat\sigma^2(p) &= \hat\sigma^2(p\,|i)\\
\hat\sigma^2(h) &= \hat\sigma^2(h|i) - \frac{\hat\sigma^2(i\!:\!h|i)}{n_i - 1}\\
\hat\sigma^2(i\!:\!h) &= \frac{\hat\sigma^2(i\!:\!h|i)}{t_i}\\
\hat\sigma^2(ph) &= \hat\sigma^2(ph|i) - \frac{\hat\sigma^2(pi\!:\!h|i)}{n_i - 1}\\
\hat\sigma^2(pi\!:\!h) &= \frac{\hat\sigma^2(pi\!:\!h|i)}{t_i}
\end{aligned}
$$

B.3 Boot–h

Table 10: Coefficients of μ^2 and Variance Components in Expected Values of T Terms for Balanced $p \times (i\!:\!h)$ Design with Boot–h

	Coefficients					
$\boldsymbol{E}T$ term	μ^2	$\sigma^2(pi{:}h)$	$\sigma^2(ph)$	$\sigma^2(i{:}h)$	$\sigma^2(h)$	$\sigma^2(p)$
$\boldsymbol{E}T(p\vert h)$	$n_p n_i n_h$	$n_p s_h$	$n_p n_i s_h$	$n_p s_h$	$n_p n_i s_h$	$n_p n_i n_h$
$\boldsymbol{E}T(h\vert h)$	$n_p n_i n_h$	n_h	$n_i n_h$	$n_p n_h$	$n_p n_i n_h$	$n_i n_h$
$\boldsymbol{E}T(i{:}h\vert h)$	$n_p n_i n_h$	$n_i n_h$	$n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_i n_h$
$\boldsymbol{E}T(ph\vert h)$	$n_p n_i n_h$	$n_p n_h$	$n_p n_i n_h$	$n_p n_h$	$n_p n_i n_h$	$n_p n_i n_h$
$\boldsymbol{E}T(pi{:}h\vert h)$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$
$\boldsymbol{E}T(\mu\vert h)$	$n_p n_i n_h$	s_h	$n_i s_h$	$n_p s_h$	$n_p n_i s_h$	$n_i n_h$

Expected Mean Square Equations

$$\begin{aligned}
\boldsymbol{E}MS(p|h) &= s_h\,\sigma^2(pi\!:\!h) + s_h n_i\,\sigma^2(ph) + n_i n_h\,\sigma^2(p)\\
\boldsymbol{E}MS(h|h) &= t_h\,\sigma^2(pi\!:\!h) + t_h n_i\,\sigma^2(ph) + t_h n_p\,\sigma^2(i\!:\!h) + t_h n_p n_i\,\sigma^2(h)\\
\boldsymbol{E}MS(i{:}h|h) &= \sigma^2(pi\!:\!h) + n_p\,\sigma^2(i\!:\!h)\\
\boldsymbol{E}MS(ph|h) &= t_h\,\sigma^2(pi\!:\!h) + t_h n_i\,\sigma^2(ph)\\
\boldsymbol{E}MS(pi{:}h|h) &= \sigma^2(pi\!:\!h)
\end{aligned}$$

Unbiased Estimators of Variance Components in terms of Bootstrap Estimators

$$\begin{aligned}
\hat\sigma^2(p) &= \hat\sigma^2(p\,|h) - \frac{\hat\sigma^2(ph|h)}{n_h-1} - \frac{\hat\sigma^2(pi\!:\!h|h)}{n_i(n_h-1)}\\
\hat\sigma^2(h) &= \frac{1}{t_h}\left[\hat\sigma^2(h|h) + \frac{\hat\sigma^2(i\!:\!h|h)}{n_i n_h}\right]\\
\hat\sigma^2(i\!:\!h) &= \hat\sigma^2(i\!:\!h|h)\\
\hat\sigma^2(ph) &= \frac{1}{t_h}\left[\hat\sigma^2(ph|h) + \frac{\hat\sigma^2(pi\!:\!h|h)}{n_i n_h}\right]\\
\hat\sigma^2(pi\!:\!h) &= \hat\sigma^2(pi\!:\!h|h)
\end{aligned}$$

B.4 Boot–p, i

Table 11: Coefficients of μ^2 and Variance Components in Expected Values of T Terms for Balanced $p \times (i\!:\!h)$ Design with Boot–p, i

	Coefficients					
$\boldsymbol{E}T$ term	μ^2	$\sigma^2(pi{:}h)$	$\sigma^2(ph)$	$\sigma^2(i{:}h)$	$\sigma^2(h)$	$\sigma^2(p)$
$\boldsymbol{E}T(p\|p,i)$	$n_pn_in_h$	n_ps_i	n_pn_i	n_ps_i	n_pn_i	$n_pn_in_h$
$\boldsymbol{E}T(h\|p,i)$	$n_pn_in_h$	$n_hs_ps_i$	$n_in_hs_p$	$n_pn_hs_i$	$n_pn_in_h$	$n_in_hs_p$
$\boldsymbol{E}T(i{:}h\|p,i)$	$n_pn_in_h$	$n_in_hs_p$	$n_in_hs_p$	$n_pn_in_h$	$n_pn_in_h$	$n_in_hs_p$
$\boldsymbol{E}T(ph\|p,i)$	$n_pn_in_h$	$n_pn_hs_i$	$n_pn_in_h$	$n_pn_hs_i$	$n_pn_in_h$	$n_pn_in_h$
$\boldsymbol{E}T(pi{:}h\|p,i)$	$n_pn_in_h$	$n_pn_in_h$	$n_pn_in_h$	$n_pn_in_h$	$n_pn_in_h$	$n_pn_in_h$
$\boldsymbol{E}T(\mu\|p,i)$	$n_pn_in_h$	s_ps_i	n_is_p	n_ps_i	n_pn_i	$n_in_hs_p$

Expected Mean Square Equations

$$
\begin{aligned}
\boldsymbol{E}MS(p|p,i) &= t_ps_i\,\sigma^2(pi\!:\!h) + t_pn_i\,\sigma^2(ph) + t_pn_in_h\,\sigma^2(p)\\
\boldsymbol{E}MS(h|p,i) &= s_ps_i\,\sigma^2(pi\!:\!h) + s_pn_i\,\sigma^2(ph) + s_in_p\,\sigma^2(i\!:\!h) + n_pn_i\,\sigma^2(h)\\
\boldsymbol{E}MS(i{:}h|p,i) &= s_pt_i\,\sigma^2(pi\!:\!h) + t_in_p\,\sigma^2(i\!:\!h)\\
\boldsymbol{E}MS(ph|p,i) &= t_ps_i\,\sigma^2(pi\!:\!h) + t_pn_i\,\sigma^2(ph)\\
\boldsymbol{E}MS(pi{:}h|p,i) &= t_pt_i\,\sigma^2(pi\!:\!h)
\end{aligned}
$$

Unbiased Estimators of Variance Components in terms of Bootstrap Estimators

$$
\begin{aligned}
\hat\sigma^2(p) &= \hat\sigma^2(p\,|p,i)\\
\hat\sigma^2(h) &= \hat\sigma^2(h|p,i) - \frac{\hat\sigma^2(i\!:\!h|p,i)}{n_i-1} - \frac{\hat\sigma^2(ph|p,i)}{n_p-1} + \frac{\hat\sigma^2(pi\!:\!h|p,i)}{(n_p-1)(n_i-1)}\\
\hat\sigma^2(i\!:\!h) &= \frac{1}{t_i}\left[\hat\sigma^2(i\!:\!h|p,i) - \frac{\hat\sigma^2(pi\!:\!h|p,i)}{n_p-1}\right]\\
\hat\sigma^2(ph) &= \frac{1}{t_p}\left[\hat\sigma^2(ph|p,i) - \frac{\hat\sigma^2(pi\!:\!h|p,i)}{n_i-1}\right]\\
\hat\sigma^2(pi\!:\!h) &= \frac{\hat\sigma^2(pi\!:\!h|p,i)}{t_pt_i}
\end{aligned}
$$

B.5 Boot–p, h

Table 12: Coefficients of μ^2 and Variance Components in Expected Values of T Terms for Balanced $p \times (i\!:\!h)$ Design with Boot–p, h

	Coefficients					
$\boldsymbol{E}T$ term	μ^2	$\sigma^2(pi{:}h)$	$\sigma^2(ph)$	$\sigma^2(i{:}h)$	$\sigma^2(h)$	$\sigma^2(p)$
$\boldsymbol{E}T(p\|p,h)$	$n_p n_i n_h$	$n_p s_h$	$n_p n_i s_h$	$n_p s_h$	$n_p n_i s_h$	$n_p n_i n_h$
$\boldsymbol{E}T(h\|p,h)$	$n_p n_i n_h$	$n_h s_p$	$n_i n_h s_p$	$n_p n_h$	$n_p n_i n_h$	$n_i n_h s_p$
$\boldsymbol{E}T(i{:}h\|p,h)$	$n_p n_i n_h$	$n_i n_h s_p$	$n_i n_h s_p$	$n_p n_i n_h$	$n_p n_i n_h$	$n_i n_h s_p$
$\boldsymbol{E}T(ph\|p,h)$	$n_p n_i n_h$	$n_p n_h$	$n_p n_i n_h$	$n_p n_h$	$n_p n_i n_h$	$n_p n_i n_h$
$\boldsymbol{E}T(pi{:}h\|p,h)$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$
$\boldsymbol{E}T(\mu\|p,h)$	$n_p n_i n_h$	$s_p s_h$	$n_i s_p s_h$	$n_p s_h$	$n_p n_i s_h$	$n_i n_h s_p$

Expected Mean Square Equations

$$
\begin{aligned}
\boldsymbol{E}MS(p|p,h) &= t_p s_h\,\sigma^2(pi\!:\!h) + t_p s_h n_i\,\sigma^2(ph) + t_p n_i n_h\,\sigma^2(p)\\
\boldsymbol{E}MS(h|p,h) &= s_p t_h\,\sigma^2(pi\!:\!h) + s_p t_h n_i\,\sigma^2(ph) + t_h n_p\,\sigma^2(i\!:\!h) + t_h n_p n_i\,\sigma^2(h)\\
\boldsymbol{E}MS(i{:}h|p,h) &= s_p\,\sigma^2(pi\!:\!h) + n_p\,\sigma^2(i\!:\!h)\\
\boldsymbol{E}MS(ph|p,h) &= t_p t_h\,\sigma^2(pi\!:\!h) + t_p t_h n_i\,\sigma^2(ph)\\
\boldsymbol{E}MS(pi{:}h|p,h) &= t_p\,\sigma^2(pi\!:\!h)
\end{aligned}
$$

Unbiased Estimators of Variance Components in terms of Bootstrap Estimators

$$
\begin{aligned}
\hat\sigma^2(p) &= \frac{1}{t_p}\left[\hat\sigma^2(p\,|p,h) - \frac{\hat\sigma^2(ph|p,h)}{n_h-1} - \frac{\hat\sigma^2(pi\!:\!h|p,h)}{n_i(n_h-1)}\right]\\
\hat\sigma^2(h) &= \frac{1}{t_h}\left[\hat\sigma^2(h|p,h) + \frac{\hat\sigma^2(i\!:\!h|p,h)}{n_i n_h} - \frac{\hat\sigma^2(ph|p,h)}{n_p-1} - \frac{\hat\sigma^2(pi\!:\!h|p,h)}{n_i n_h(n_p-1)}\right]\\
\hat\sigma^2(i\!:\!h) &= \hat\sigma^2(i\!:\!h|p,h) - \frac{\hat\sigma^2(pi\!:\!h|p,h)}{n_p-1}\\
\hat\sigma^2(ph) &= \frac{1}{t_p t_h}\left[\hat\sigma^2(ph|p,h) + \frac{\hat\sigma^2(pi\!:\!h|p,h)}{n_i n_h}\right]\\
\hat\sigma^2(pi\!:\!h) &= \frac{\hat\sigma^2(pi\!:\!h|p,h)}{t_p}
\end{aligned}
$$

B.6 Boot–i, h

Table 13: Coefficients of μ^2 and Variance Components in Expected Values of T Terms for Balanced $p \times (i\!:\!h)$ Design with Boot–i, h

	Coefficients					
$\boldsymbol{E}T$ term	μ^2	$\sigma^2(pi{:}h)$	$\sigma^2(ph)$	$\sigma^2(i{:}h)$	$\sigma^2(h)$	$\sigma^2(p)$
$\boldsymbol{E}T(p\|i,h)$	$n_p n_i n_h$	$n_p s_i s_h$	$n_p n_i s_h$	$n_p s_i s_h$	$n_p n_i s_h$	$n_p n_i n_h$
$\boldsymbol{E}T(h\|i,h)$	$n_p n_i n_h$	$n_h s_i$	$n_i n_h$	$n_p n_h s_i$	$n_p n_i n_h$	$n_i n_h$
$\boldsymbol{E}T(i{:}h\|i,h)$	$n_p n_i n_h$	$n_i n_h$	$n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_i n_h$
$\boldsymbol{E}T(ph\|i,h)$	$n_p n_i n_h$	$n_p n_h s_i$	$n_p n_i n_h$	$n_p n_h s_i$	$n_p n_i n_h$	$n_p n_i n_h$
$\boldsymbol{E}T(pi{:}h\|i,h)$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$	$n_p n_i n_h$
$\boldsymbol{E}T(\mu\|i,h)$	$n_p n_i n_h$	$s_i s_h$	$n_i s_h$	$n_p s_i s_h$	$n_p n_i s_h$	$n_i n_h$

Expected Mean Square Equations

$$\begin{aligned}
\boldsymbol{E}MS(p|i,h) &= s_i s_h\,\sigma^2(pi\!:\!h) + s_h n_i\,\sigma^2(ph) + n_i n_h\,\sigma^2(p)\\
\boldsymbol{E}MS(h|i,h) &= s_i t_h\,\sigma^2(pi\!:\!h) + t_h n_i\,\sigma^2(ph) + s_i t_h n_p\,\sigma^2(i\!:\!h) + t_h n_p n_i\,\sigma^2(h)\\
\boldsymbol{E}MS(i{:}h|i,h) &= t_i\,\sigma^2(pi\!:\!h) + t_i n_p\,\sigma^2(i\!:\!h)\\
\boldsymbol{E}MS(ph|i,h) &= s_i t_h\,\sigma^2(pi\!:\!h) + t_h n_i\,\sigma^2(ph)\\
\boldsymbol{E}MS(pi{:}h|i,h) &= t_i\,\sigma^2(pi\!:\!h)
\end{aligned}$$

Unbiased Estimators of Variance Components in terms of Bootstrap Estimators

$$\begin{aligned}
\hat\sigma^2(p) &= \hat\sigma^2(p\,|i,h) - \frac{\hat\sigma^2(ph|i,h)}{n_h-1} - \frac{\hat\sigma^2(pi\!:\!h|i,h)}{n_i(n_h-1)}\\
\hat\sigma^2(h) &= \frac{1}{t_h}\left[\hat\sigma^2(h|i,h) - \left(\frac{s_i t_h - t_i}{n_i-1}\right)\hat\sigma^2(i\!:\!h|i,h)\right]\\
\hat\sigma^2(i\!:\!h) &= \frac{\hat\sigma^2(i\!:\!h|i,h)}{t_i}\\
\hat\sigma^2(ph) &= \frac{1}{t_h}\left[\hat\sigma^2(ph|i,h) - \left(\frac{s_i t_h - t_i}{n_i-1}\right)\hat\sigma^2(pi\!:\!h|i,h)\right]\\
\hat\sigma^2(pi\!:\!h) &= \frac{\hat\sigma^2(pi\!:\!h|i,h)}{t_i}
\end{aligned}$$

B.7 Boot–p, i, h

Table 14: Coefficients of μ^2 and Variance Components in Expected Values of T Terms for Balanced $p \times (i\!:\!h)$ Design with Boot–p, i, h

	Coefficients					
$\boldsymbol{E}T$ term	μ^2	$\sigma^2(pi\!:\!h)$	$\sigma^2(ph)$	$\sigma^2(i\!:\!h)$	$\sigma^2(h)$	$\sigma^2(p)$
$\boldsymbol{E}T(p\vert p,i,h)$	$n_pn_in_h$	$n_ps_is_h$	$n_pn_is_h$	$n_ps_is_h$	$n_pn_is_h$	$n_pn_in_h$
$\boldsymbol{E}T(h\vert p,i,h)$	$n_pn_in_h$	$n_hs_ps_i$	$n_in_hs_p$	$n_pn_hs_i$	$n_pn_in_h$	$n_in_hs_p$
$\boldsymbol{E}T(i\!:\!h\vert p,i,h)$	$n_pn_in_h$	$n_in_hs_p$	$n_in_hs_p$	$n_pn_in_h$	$n_pn_in_h$	$n_in_hs_p$
$\boldsymbol{E}T(ph\vert p,i,h)$	$n_pn_in_h$	$n_pn_hs_i$	$n_pn_in_h$	$n_pn_hs_i$	$n_pn_in_h$	$n_pn_in_h$
$\boldsymbol{E}T(pi\!:\!h\vert p,i,h)$	$n_pn_in_h$	$n_pn_in_h$	$n_pn_in_h$	$n_pn_in_h$	$n_pn_in_h$	$n_pn_in_h$
$\boldsymbol{E}T(\mu\vert p,i,h)$	$n_pn_in_h$	$s_ps_is_h$	$n_is_ps_h$	$n_ps_is_h$	$n_pn_is_h$	$n_in_hs_p$

Expected Mean Square Equations

$$
\begin{aligned}
\boldsymbol{E}MS(p\,|p,i,h) &= t_ps_is_h\,\sigma^2(pi\!:\!h) + t_ps_hn_i\,\sigma^2(ph) + t_pn_in_h\,\sigma^2(p)\\
\boldsymbol{E}MS(h|p,i,h) &= s_ps_it_h\,\sigma^2(pi\!:\!h) + s_pt_hn_i\,\sigma^2(ph)\\
&\quad + s_it_hn_p\,\sigma^2(i\!:\!h) + t_hn_pn_i\,\sigma^2(h)\\
\boldsymbol{E}MS(i\!:\!h|p,i,h) &= s_pt_i\,\sigma^2(pi\!:\!h) + t_in_p\,\sigma^2(i\!:\!h)\\
\boldsymbol{E}MS(ph|p,i,h) &= t_ps_it_h\,\sigma^2(pi\!:\!h) + t_pt_hn_i\,\sigma^2(ph)\\
\boldsymbol{E}MS(pi\!:\!h|p,i,h) &= t_pt_i\,\sigma^2(pi\!:\!h)
\end{aligned}
$$

Unbiased Estimators of Variance Components in terms of Bootstrap Estimators

$$
\begin{aligned}
\hat\sigma^2(p) &= \frac{1}{t_p}\left[\hat\sigma^2(p\,|p,i,h) - \frac{\hat\sigma^2(ph|p,i,h)}{n_h-1} - \frac{\hat\sigma^2(pi\!:\!h|p,i,h)}{n_i(n_h-1)}\right]\\
\hat\sigma^2(h) &= \frac{1}{t_h}\left\{\hat\sigma^2(h|p,i,h) - \frac{\hat\sigma^2(ph|p,i,h)}{n_p-1}\right.\\
&\quad\left. - \left(\frac{s_it_h-t_i}{n_i-1}\right)\left[\hat\sigma^2(i\!:\!h|p,i,h) - \frac{\hat\sigma^2(pi\!:\!h|p,i,h)}{n_p-1}\right]\right\}\\
\hat\sigma^2(i\!:\!h) &= \frac{1}{t_i}\left[\hat\sigma^2(i\!:\!h|p,i,h) - \frac{\hat\sigma^2(pi\!:\!h|p,i,h)}{n_p-1}\right]\\
\hat\sigma^2(ph) &= \frac{1}{t_pt_h}\left[\hat\sigma^2(ph|p,i,h) - \left(\frac{s_it_h-t_i}{n_i-1}\right)\hat\sigma^2(pi\!:\!h|p,i,h)\right]\\
\hat\sigma^2(pi\!:\!h) &= \frac{\hat\sigma^2(pi\!:\!h|p,i,h)}{t_pt_i}
\end{aligned}
$$

C Bias-Corrected Estimates of Variance Components for the $p \times i \times h$ Design

C.1 Boot-p

$$
\begin{aligned}
\hat{\sigma}^2(p) &= \frac{\hat{\sigma}^2(p|p)}{t_p} \\
\hat{\sigma}^2(i) &= \hat{\sigma}^2(i|p) - \frac{\hat{\sigma}^2(pi|p)}{n_p - 1} \\
\hat{\sigma}^2(h) &= \hat{\sigma}^2(h|p) - \frac{\hat{\sigma}^2(ph|p)}{n_p - 1} \\
\hat{\sigma}^2(pi) &= \frac{\hat{\sigma}^2(pi|p)}{t_p} \\
\hat{\sigma}^2(ph) &= \frac{\hat{\sigma}^2(ph|p)}{t_p} \\
\hat{\sigma}^2(ih) &= \hat{\sigma}^2(ih|p) - \frac{\hat{\sigma}^2(pih|p)}{n_p - 1} \\
\hat{\sigma}^2(pih) &= \frac{\hat{\sigma}^2(pih|p)}{t_p}
\end{aligned}
$$

C.2 Boot-i

$$
\begin{aligned}
\hat{\sigma}^2(p) &= \hat{\sigma}^2(p|i) - \frac{\hat{\sigma}^2(pi|i)}{n_i - 1} \\
\hat{\sigma}^2(i) &= \frac{\hat{\sigma}^2(i|i)}{t_i} \\
\hat{\sigma}^2(h) &= \hat{\sigma}^2(h|i) - \frac{\hat{\sigma}^2(ih|i)}{n_i - 1} \\
\hat{\sigma}^2(pi) &= \frac{\hat{\sigma}^2(pi|i)}{t_i} \\
\hat{\sigma}^2(ph) &= \hat{\sigma}^2(ph|i) - \frac{\hat{\sigma}^2(pih|i)}{n_i - 1} \\
\hat{\sigma}^2(ih) &= \frac{\hat{\sigma}^2(ih|i)}{t_i} \\
\hat{\sigma}^2(pih) &= \frac{\hat{\sigma}^2(pih|i)}{t_i}
\end{aligned}
$$

C.3 Boot-h

$$
\begin{aligned}
\hat{\sigma}^2(p) &= \hat{\sigma}^2(p|h) - \frac{\hat{\sigma}^2(ph|h)}{n_h - 1} \\
\hat{\sigma}^2(i) &= \hat{\sigma}^2(i|h) - \frac{\hat{\sigma}^2(ih|h)}{n_h - 1} \\
\hat{\sigma}^2(h) &= \frac{\hat{\sigma}^2(h|h)}{t_h} \\
\hat{\sigma}^2(pi) &= \hat{\sigma}^2(pi|h) - \frac{\hat{\sigma}^2(pih|h)}{n_h - 1} \\
\hat{\sigma}^2(ph) &= \frac{\hat{\sigma}^2(ph|h)}{t_h} \\
\hat{\sigma}^2(ih) &= \frac{\hat{\sigma}^2(ih|h)}{t_h} \\
\hat{\sigma}^2(pih) &= \frac{\hat{\sigma}^2(pih|h)}{t_h}
\end{aligned}
$$

C.4 Boot-p, i

$$
\begin{aligned}
\hat{\sigma}^2(p) &= \frac{1}{t_p}\left[\hat{\sigma}^2(p|p,i) - \frac{\hat{\sigma}^2(pi|p,i)}{n_i - 1}\right] \\
\hat{\sigma}^2(i) &= \frac{1}{t_i}\left[\hat{\sigma}^2(i|p,i) - \frac{\hat{\sigma}^2(pi|p,i)}{n_p - 1}\right] \\
\hat{\sigma}^2(h) &= \hat{\sigma}^2(h|p,i) - \frac{\hat{\sigma}^2(ph|p,i)}{n_p - 1} - \frac{\hat{\sigma}^2(ih)|p,i)}{n_i - 1} + \frac{\hat{\sigma}^2(pih|p,i)}{(n_p - 1)(n_i - 1)} \\
\hat{\sigma}^2(pi) &= \frac{\hat{\sigma}^2(pi|p,i)}{t_p t_i} \\
\hat{\sigma}^2(ph) &= \frac{1}{t_p}\left[\hat{\sigma}^2(ph|p,i) - \frac{\hat{\sigma}^2(pih|p,i)}{n_i - 1}\right] \\
\hat{\sigma}^2(ih) &= \frac{1}{t_i}\left[\hat{\sigma}^2(ih|p,i) - \frac{\hat{\sigma}^2(pih|p,i)}{n_p - 1}\right] \\
\hat{\sigma}^2(pih) &= \frac{\hat{\sigma}^2(pih|p,i)}{t_p t_i}
\end{aligned}
$$

C.5 Boot-p, h

$$\begin{aligned}
\hat{\sigma}^2(p) &= \frac{1}{t_p}\left[\hat{\sigma}^2(p|p,h) - \frac{\hat{\sigma}^2(ph|p,h)}{n_h - 1}\right] \\
\hat{\sigma}^2(i) &= \hat{\sigma}^2(i|p,h) - \frac{\hat{\sigma}^2(pi|p,h)}{n_p - 1} - \frac{\hat{\sigma}^2(ih)|p,h)}{n_h - 1} + \frac{\hat{\sigma}^2(pih|p,h)}{(n_p - 1)(n_h - 1)} \\
\hat{\sigma}^2(h) &= \frac{1}{t_h}\left[\hat{\sigma}^2(h|p,h) - \frac{\hat{\sigma}^2(ph|p,h)}{n_p - 1}\right] \\
\hat{\sigma}^2(pi) &= \frac{1}{t_p}\left[\hat{\sigma}^2(ph|p,h) - \frac{\hat{\sigma}^2(pih|p,h)}{n_h - 1}\right] \\
\hat{\sigma}^2(ph) &= \frac{\hat{\sigma}^2(ph|p,h)}{t_p t_h} \\
\hat{\sigma}^2(ih) &= \frac{1}{t_h}\left[\hat{\sigma}^2(ih|p,h) - \frac{\hat{\sigma}^2(pih|p,h)}{n_p - 1}\right] \\
\hat{\sigma}^2(pih) &= \frac{\hat{\sigma}^2(pih|p,h)}{t_p t_h}
\end{aligned}$$

C.6 Boot-i, h

$$\begin{aligned}
\hat{\sigma}^2(p) &= \hat{\sigma}^2(p|i,h) - \frac{\hat{\sigma}^2(pi|i,h)}{n_p - 1} - \frac{\hat{\sigma}^2(ph)|i,h)}{n_h - 1} + \frac{\hat{\sigma}^2(pih|i,h)}{(n_i - 1)(n_h - 1)} \\
\hat{\sigma}^2(i) &= \frac{1}{t_i}\left[\hat{\sigma}^2(i|i,h) - \frac{\hat{\sigma}^2(ih|i,h)}{n_h - 1}\right] \\
\hat{\sigma}^2(h) &= \frac{1}{t_h}\left[\hat{\sigma}^2(h|i,h) - \frac{\hat{\sigma}^2(ph|i,h)}{n_i - 1}\right] \\
\hat{\sigma}^2(pi) &= \frac{1}{t_i}\left[\hat{\sigma}^2(ph|i,h) - \frac{\hat{\sigma}^2(pih|i,h)}{n_h - 1}\right] \\
\hat{\sigma}^2(ph) &= \frac{1}{t_h}\left[\hat{\sigma}^2(ph|i,h) - \frac{\hat{\sigma}^2(pih|i,h)}{n_i - 1}\right] \\
\hat{\sigma}^2(ih) &= \frac{\hat{\sigma}^2(ih|i,h)}{t_i t_h} \\
\hat{\sigma}^2(pih) &= \frac{\hat{\sigma}^2(pih|i,h)}{t_i t_h}
\end{aligned}$$

C.7 Boot-p, i, h

$$
\begin{aligned}
\hat{\sigma}^2(p) &= \frac{1}{t_p}\left[\hat{\sigma}^2(p|p,i,h) - \frac{\hat{\sigma}^2(pi|p,i,h)}{n_i - 1} - \frac{\hat{\sigma}^2(ph|p,i,h)}{n_h - 1} + \frac{\hat{\sigma}^2(pih|p,i,h)}{(n_i - 1)(n_h - 1)}\right] \\
\hat{\sigma}^2(i) &= \frac{1}{t_i}\left[\hat{\sigma}^2(i|p,i,h) - \frac{\hat{\sigma}^2(pi|p,i,h)}{n_p - 1} - \frac{\hat{\sigma}^2(ih|p,i,h)}{n_h - 1} + \frac{\hat{\sigma}^2(pih|p,i,h)}{(n_p - 1)(n_h - 1)}\right] \\
\hat{\sigma}^2(h) &= \frac{1}{t_h}\left[\hat{\sigma}^2(h|p,i,h) - \frac{\hat{\sigma}^2(ph|p,i,h)}{n_p - 1} - \frac{\hat{\sigma}^2(ih|p,i,h)}{n_i - 1} + \frac{\hat{\sigma}^2(pih|p,i,h)}{(n_p - 1)(n_i - 1)}\right] \\
\hat{\sigma}^2(pi) &= \frac{1}{t_p t_i}\left[\hat{\sigma}^2(pi|p,i,h) - \frac{\hat{\sigma}^2(pih|p,i,h)}{n_h - 1}\right] \\
\hat{\sigma}^2(ph) &= \frac{1}{t_p t_h}\left[\hat{\sigma}^2(ph|p,i,h) - \frac{\hat{\sigma}^2(pih|p,i,h)}{n_i - 1}\right] \\
\hat{\sigma}^2(ih) &= \frac{1}{t_i t_h}\left[\hat{\sigma}^2(ih|p,i,h) - \frac{\hat{\sigma}^2(pih|p,i,h)}{n_p - 1}\right] \\
\hat{\sigma}^2(pih) &= \frac{\hat{\sigma}^2(pih|p,i,h)}{t_p t_i t_h}
\end{aligned}
$$